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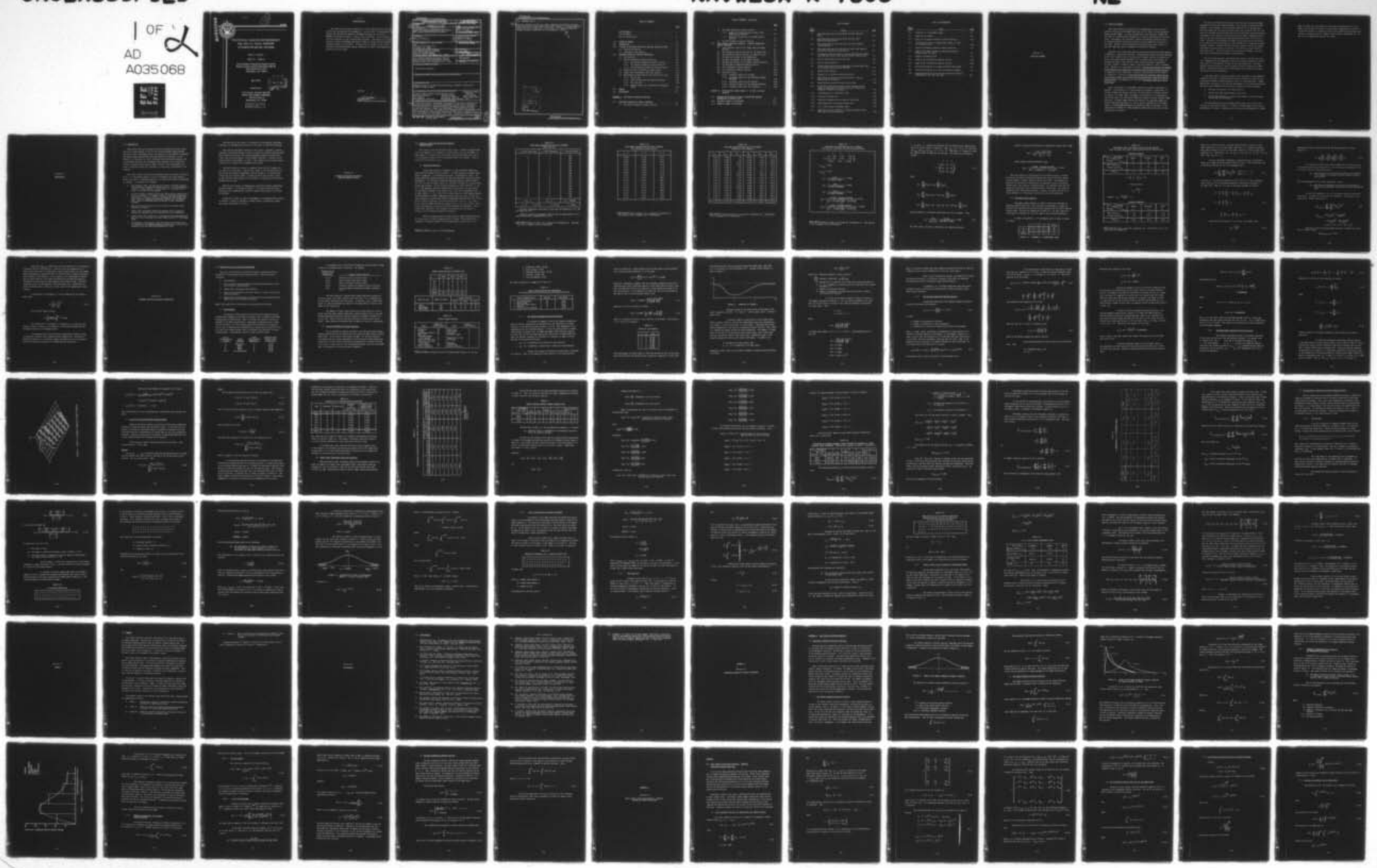

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STATISTICAL ANALYSIS METHODOLOGY FOR THE U.S. NAVY AIRCREW AUTOMATED ESCAPE SYSTEMS

Henry L. Crowson

CACI, Inc. - Federal

Naval Weapons Engineering Support Activity
Weapon Systems Analysis Department (ESA-19)
Washington Navy Yard
Washington, D.C. 20374

MAY 1976

Prepared for

Crew Systems Division (AR-531)
NAVAL AIR SYSTEMS COMMAND
Jefferson Plaza 2
Washington, D.C. 20361

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D. S. Watson

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A statistical analysis methodology is proposed for the study of Aircrew Automated Escape Systems. A detailed explanation is given for the following: (1) data analysis; (2) applicable discrete probability density functions to include the binomial, Poisson, and multinomial; (3) Bayesian statistical theory and applications; (4) higher order contingency table data analysis; and (5) non-parametric (distribution free) statistical tests, to include the run test, the trend test, and Fisher's Exact Test. Continuous probability density → next page

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functions discussed include the normal, gamma and beta density functions. A brief exposition of small sample statistical analysis, to include sampling probability density functions, is also given. The sampling probability density functions considered are the Chi-Square, t, and F.

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Section 1.0

EXECUTIVE SUMMARY

1.0 Executive Summary

This report presents an approach to a methodology for the statistical analysis of ejection performance data from the U.S. Navy's Aircrew Automated Escape Systems (AAES). A central problem area in any analysis of AAES ejection data is a comprehensive examination of injuries associated with the ejections. Accordingly, the statistical analysis methodology developed herein is oriented primarily toward an in-depth critical examination of aircrew injury data. However, the methodology can be applied with equal facility to any other AAES data which is amenable to statistical analysis.

Basically, this report initially surveys selected analyses already applied to AAES data. Results are discussed. Although many reports have been developed in the area of AAES studies, the particular document selected for examination here is the report dated 8 November 1974, "An Evaluation of the Effects of Spreader Gun Usage in Navy Ejection Seat Personnel Parachute Subsystems," by Messrs. Robert L. Wallace, James W. Pope and Frederick C. Guill of NAVAIR 53121. Appendix I in that report, entitled, "Statistical Evaluation of ESCAPAC Series Ejection Seat Components' Correlation with Incidence of Neck Injuries," is comprised of four subsections. Specific attention here will be devoted to subsection 3, entitled, "Correlation Analysis of Potential Causal Factors for Increase in Neck Injuries Sustained During Ejections Using HS-1A (RA-5C) Ejection Seats," and subsection 4, entitled, "Contingency Analysis of RA-5C Escape Data."

Next, an approach to a reasonable statistical analysis methodology is proposed. This consists of (1) data analysis (re-format, if necessary), (2) discrete probability density functions, (3) application of Bayesian statistical theory, (4) higher order contingency table analysis, (5) non-parametric statistical tests, (6) continuous probability density functions (normal, gamma and beta), (7) small sample statistical analysis, (8) sampling probability density functions (Chi-square, t, and F), and (9) application of confidence limits. A bibliography pertinent to statistical ideas is included. A few references to selected U.S. Navy documents studied during the course of this project also are listed.

The lack of systematic processes for collection and evaluation of AAES operating statistics has hampered NAVAIR in its response to developing AAES problems. As an example, a study recently completed on the spreader gun used in an AAES (see the report mentioned above) was initiated after the discovery by the Naval Aviation Safety Center of a significant difference between the number of neck injuries occurring with various types of ESCAPAC ejection seats in the A-4 aircraft and the HS-1/HS-1A ejection seats in the A-5 aircraft. Initially, the difference in number of neck injuries was attributed by the Naval Aviation Safety Center to the spreader gun. Subsequent in-depth analyses, however, revealed that neck injuries, among A-4 users, were highly correlated with the presence, or absence, of a ballistic powered haulback type inertia reel. In A-5 aircraft, neck injuries were thought to be correlated with the malfunction of another system element.

The neck injury analysis, and similar past studies, indicated the need for a comprehensive and independent study of all ejection systems to evaluate their quality, effectiveness, reliability and maintainability. Moreover, analysis of past testing, in view of current in-service experience in AAES quality, would help determine what relationships exist between test results and actual in-service experience.

A one-time study of existing systems is not sufficient to solve NAVAIR's problem in this area. A long-term solution must include the collection and processing by NAVAIR of AAES performance data on a continuing basis. Complimenting data collection is the need for development of an analysis methodology which will assist NAVAIR with its mission in keeping with the following:

- Efficient allocation of its AAES resources
- Provide more AAES responsiveness to the Fleet
- Resolve AAES problems in a manner that enhances in-service system reliability and performance.

To assist NAVAIR with selected AAES problem areas, this study is constructed in four distinct phases: (1) Phase I, establishment of an analysis methodology, (2) Phase II, application of the analysis methodology to a single

AAES, (3) Phase III, application of the analysis methodology to a set of AAES's and (4) Phase IV, (based on results and recommendations in Phases I, II and III) institute a scheme which will monitor AAES production/test activities on a continuing basis. Proposed future efforts under Phases II, III and IV of the AAES study conclude this report.

Section 2.0

INTRODUCTION

2.0 Introduction

Many reports and other documents have been developed by various organizations to study problems in the area of Aircrew Automated Escape Systems. One recent report (8 November 1974), "An Evaluation of the Effects of Spreader Gun Usage in Navy Ejection Seat Personnel Parachute Subsystems," by R.L. Wallace, J.W. Pope and F.C. Guill, confirmed that statistical analyses, such as correlation analysis and contingency table analysis, can be used effectively to analyze problem areas in AAES. Central to the investigative efforts in the above report was an analysis of injury data, specifically, neck injuries.

This report surveys two statistical methods used in the report by Wallace, et al: (1) correlation analysis, and (2) contingency table data analysis. After that survey, a reasonable approach to a statistical analysis methodology is proposed. It includes the following:

- Data Analysis--Here, the data are to be given a multiple classification scheme in consonance with MORs such as 1 - no injury/minimal injury, 2 - minor injury, 3 - major injury, 4 - fatality, and 5 - other, to include lost and unknowns.
- Discrete Probability Density Functions--Three discrete density functions were chosen: (1) binomial, useful in analyzing dichotomous situations, (2) Poisson, useful in attacking special binomial problems, as well as analyzing reliability problems, and (3) multinomial, useful in analyzing events having two or more independent outcomes.
- Bayesian Statistical Theory--This is useful in refining a priori probability estimates.
- Higher Order Contingency Table Data Analysis--This is useful in testing the hypothesis that two characteristics are independent.
- Fisher's Exact Test--This test is a method for obtaining exact probabilities of the occurrence of events among entries in a contingency table.
- Non-Parametric (Distribution Free) Statistical Tests--These tests are designed to tell whether phenomena represented by the data occur randomly or have an underlying deterministic trend.

The main body of the report is concluded by a bibliography containing references in the AAES area as well as in the field of statistical analysis.

There are two appendixes attached to this report: Appendix A contains additional statistical analyses, such as a survey of selected continuous probability density functions, and selected sampling probability density functions, such as the Chi-square, t and F density functions. A survey of confidence limits, with examples, is given. This enables one to attach a given confidence to a probabilistic statement that is made.

Appendix B contains a survey of proposed future efforts in Phases II, III, and IV of this project. Specifically, Phase II will be application of the statistical analysis methodology, derived in Phase I, to actual operational data for a specific AAES. A single attribute data stream, as well as dual attribute data streams, will be studied. An analysis of the results obtained will be given.

Phase III will consist of applying the statistical analysis methodology derived in Phase I, as refined by Phase II, to a class of ejection seats. Conclusions and recommendations would be a natural end product of the Phase III analyais.

Activities in Phase IV, that is development of an AAES monitoring system, is sketched. Details of procedure in Phase IV must, of necessity, depend upon output from Phases II and III.

Section 3.0

A SURVEY OF SELECTED STATISTICAL ANALYSES APPLIED TO DATE

3.0 A Survey of Selected Statistical Analyses Applied to Date

Basic material for the discussion that follows is found in sub-sections 3 and 4 of Appendix I in reference 13, specifically, [13, pp 235-279]*. In these pages it is evident that two primary statistical analyses were used: (1) correlation analysis, and (2) contingency table data analysis. Each of these will be discussed in turn.

3.1 Correlation Analysis

Injury data analysis in Appendix I of the referenced report has been restricted largely to neck injuries. In those analyses, a binary designation is employed: 1--indicates neck injury, 0--indicates no injury. No parent probability density function has been associated with the given binary designation. To illustrate correlation analyses employed, consider Tables 3-1, 3-2, 3-3 and 3-4 found in [13, pp. 273-276]. In Table 3-1, several observations can be made: (1) the relative frequency of a neck injury [5, p. 36; 11, p. 99] sometimes defined as probability of occurrence of a neck injury, for the sample displayed in the table is $P_y = 3/17$; (2) the probability of no neck injury is $1 - P_y = 14/17$; (3) all neck injuries in the sample examined occurred whenever the HS-1A (upgraded) ejection seat was used; (4) the relative frequency of ejection with an HS-1A (upgraded) ejection seat is $P_x = 8/17$; and (5) relative frequency of ejection with an HS-1 seat is $1 - P_x = 9/17$. Here it must be emphasized that the sample size (n = 17) is very small, hence little confidence can be placed in the above probability statements. Additional data, hopefully, with strength the confidence one can invest in validity of the preceding probability statements.

Relative frequency was the measure used to compute mean values for: (1) neck injury, (2) seat designation, and (3) speed. This enabled the investigator to construct Table 3-2 containing numerical values for $y - \bar{y}$,

*Numbers in brackets refer to the Bibliography.

TABLE 3-1*

HIGH SPEED EJECTIONS FROM THE RA-5C AIRCRAFT
(Observed Data)

y = Neck Injury Data	x = Seat Designation	s = Aircraft Speed
1	1	260
0	1	260
1	1	400
0	1	400
0	1	300
0	1	300
1	1	400
0	1	400
0	0	200
0	0	200
0	0	220
0	0	220
0	0	450
0	0	200
0	0	230
0	0	230
0	0	230
$\Sigma y_i = 3$	$\Sigma x_i = 8$	$\Sigma s_i = 4900.00$

$n = 17$ $\bar{x} = 0.471$ $\bar{y} = 0.1765$ $\bar{s} = 288.2$

Ejectees sustaining a neck injury of any type are denoted by a "1"; and all others by a "0".

Ejectees using HS-1A (upgraded) ejection seat are identified by a "1"; those using the HS-1 (unmodified) by a "0".

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*This table is shown as Table I-14, on page 273 in Reference 13. (See Section 6 for a complete list of references.)

TABLE 3-2*

HIGH SPEED EJECTIONS FROM THE RA-5C AIRCRAFT
(Data Modified by Mean Values)

$y_i - \bar{y}$	$x_i - \bar{x}$	$s_i - \bar{s}$
+0.824	0.529	-28.2
-0.177	0.529	-28.2
+0.824	0.529	111.8
-0.177	0.529	111.8
-0.177	0.529	11.8
-0.177	0.529	11.8
+0.824	0.529	111.8
-0.177	0.529	111.8
-0.177	-0.471	-88.2
-0.177	-0.471	-88.2
-0.177	-0.471	-68.2
-0.177	-0.471	-68.2
-0.177	-0.471	161.8
-0.177	-0.471	-88.2
-0.177	-0.471	-58.2
-0.177	-0.471	-58.2
-0.177	-0.471	-58.2

*This table is shown as Table I-15, on page 274 in Reference 13. (See Section 6 for a complete list of references.)

TABLE 3-3*

HIGH SPEED EJECTIONS FROM THE RA-5C AIRCRAFT
(Elements in the Q-Matrix)

q_{yy}	q_{xx}	q_{ss}	q_{yx}	q_{ys}	q_{xs}
0.678	0.280	795	+0.436	-23.1	-14.8
0.031	0.280	795	-0.094	+5.0	-14.8
0.678	0.280	12,544	+0.436	+92.3	+59.2
0.031	0.280	12,544	-0.094	-19.8	+59.2
0.031	0.280	144	-0.094	-2.1	+6.3
0.031	0.280	144	-0.094	-2.1	+6.3
0.678	0.280	12,544	+0.436	+92.3	+59.2
0.031	0.280	12,544	-0.094	-19.8	+59.2
0.031	0.222	7,744	0.083	+15.6	+41.4
0.031	0.222	7,744	0.083	+15.6	+41.4
0.031	0.222	4,624	0.083	+12.0	+32.0
0.031	0.222	4,624	0.083	+12.0	+32.0
0.031	0.222	26,244	0.083	-28.7	-76.3
0.031	0.222	7,744	0.083	+15.6	+41.2
0.031	0.222	3,364	0.083	+10.3	+27.3
0.031	0.222	3,364	0.083	+10.3	+27.3
0.031	0.222	3,364	0.083	+10.3	+27.3
Σ = 2.469	Σ = 4.238	Σ = 120,870	Σ = 2.525	Σ = 183.7	Σ = 413.4

*This table is shown as Table I-16 on page 275 in Reference 13. (See Section 6 for a complete list of references.)

TABLE 3-4*

HIGH SPEED EJECTIONS FROM THE RA-5C AIRCRAFT
(The Q-Matrix and Partial Correlation Coefficients)

$$Q = \begin{array}{c} \\ y \\ x \\ s \end{array} \begin{array}{c} \begin{array}{ccc} y & x & s \end{array} \\ \begin{pmatrix} 2.469 & 2.525 & 183.7 \\ 2.525 & 4.238 & 413.4 \\ 183.7 & 413.4 & 120{,}870 \end{pmatrix} \end{array}$$

$$r_{crit_{95}} = 0.497$$

$$r_{crit_{99}} = 0.623$$

$$r_{yx} = \frac{2.525}{\sqrt{(2.469)\ (4.238)}} = 0.7806$$

$$r_{ys} = \frac{183.7}{\sqrt{(2.469)\ (120{,}870)}} = 0.3363$$

$$r_{xs} = \frac{413.4}{\sqrt{(4.238)\ (120{,}870)}} = 0.5776$$

$$r_{yx.s} = \frac{0.7806 - (0.3363)\ (0.5776)}{\left[\sqrt{1 - (0.3363)^2}\right]\left[\sqrt{1 - (0.5776)^2}\right]} = 0.7627$$

$$r_{ys.x} = \frac{0.3363 - (0.7806)\ (0.5776)}{\left[\sqrt{1 - (0.7806)^2}\right]\left[\sqrt{1 - (0.5776)^2}\right]} = -\ 0.2246$$

*This table is shown as Table I-17 on page 276 in Reference 13. (See Section 6 for a complete list of references.)

$x - \bar{x}$, and $s - \bar{s}$. Values for insertion into the Q-matrix were computed from Table 3-2 and are listed in Table 3-3. Thus, $q_{yy} = (y - \bar{y})\ (y - \bar{y})$, similarly for q_{xx}, and q_{ss}, and $q_{yx} = (y - \bar{y})\ (x - \bar{x})$. From Table 3-3, numerical values for insertion into the Q-matrix were derived. The Q-matrix is defined as follows:

$$Q = \begin{array}{c} \\ y \\ x \\ s \end{array} \begin{array}{c} \begin{array}{ccc} y & x & s \end{array} \\ \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \end{array} \qquad (3.1)$$

where

$$q_{11} = \sum_{i=1}^{17} (q_{yy})_i;\ q_{12} = \sum_{i=1}^{17} (q_{yx})_i;$$

$$q_{13} = \sum_{i=1}^{17} (q_{ys})_i;\ q_{21} = q_{12};\ q_{22} = \sum_{i-1}^{17} (q_{xx})_i;$$

$$q_{23} = \sum_{i=1}^{17} (q_{xs})_i;\ q_{31} = q_{13};\ q_{32} = q_{23};\ \text{and}\ q_{33} = \sum_{i=1}^{17} (q_{ss})_i.$$

From the Q-matrix, correlation coefficients are easy to compute. Thus,

$$r_{yx} = \frac{q_{12}}{\sqrt{q_{11}\ q_{22}}} = \frac{2.525}{\sqrt{(2.469)\ (4.238)}} = 0.7806 \qquad (3.2)$$

The other linear correlation coefficients are computed similarly.

Partial correlation coefficients are computed as follows: [4, p 496]

$$r_{yx|s} = \frac{r_{yx} - (r_{ys})(r_{xs})}{\sqrt{(1 - r_{ys}^2)(1 - r_{xs}^2)}} \tag{3.3}$$

Insert numbers already developed to get:

$$r_{yx|s} = \frac{0.7806 - (0.3363)(0.5776)}{\sqrt{(1 - (0.3363)^2)(1 - (0.5776)^2)}} = 0.7627$$

This last equation is interpreted from correlation theory as follows: The partial correlation coefficient between neck injuries and seat configuration knowing, or given, speed at ejection is 0.7627. No explanation is given for the necessity or desirability of computing critical correlation coefficients, other than as a threshold value. No references were detected giving the statistical theory underlying these coefficients. No method for computing these coefficients was given. Further, the application of correlation analysis to a dichotomous situation also has been questioned.

3.2 Contingency Table Analysis

Contingency Table Analysis is another statistical technique for analyzing ejection seat data. This technique was employed on a sample of ejection data from RA-5C aircraft which were equipped with the HS-1A (updated) ejection seat. The data are summarized in Table 3-5. The data sample discussed below is believed not to be the same as that discussed previously.

A simple, but general, 2 x 2 contingency table is shown in Figure 3-1 below.

	I	II	TOTALS
A	a_1	a_2	N_a
B	b_1	b_2	N_b
TOTALS	N_1	N_2	N

FIGURE 3-1. A GENERAL 2 x 2 CONTINGENCY TABLE

TABLE 3-5*

CONTINGENCY TABLE DATA ANALYSIS FOR THE RA-5C AIRCRAFT
(Neck Injury/No Neck Injury Versus Updated Seat/No Updated Seat)

OBSERVED FREQUENCY

Seat Status → / ↓ Injury	Update f_{i1}	No Update f_{i2}	TOTAL n_i
Neck Injury f_{1j}	5	2	7
No Neck Injury f_{2j}	5	38	43
TOTAL n_j	10	40	50

$$n = \sum_i n_i = \sum_j n_j = 50$$

Expected Values

$$e_{ij} = \frac{n_i n_j}{n}$$

Example: $e_{11} = \frac{(7)\ (10)}{50} = 1.4$

EXPECTED FREQUENCY

Seat Status → / ↓ Injury	Update e_{i1}	No Update e_{i2}	TOTAL n_i
Neck Injury e_{1j}	1.4	5.6	7.0
No Neck Injury e_{2j}	8.6	34.4	43.0
TOTAL n_j	10.0	40.0	50.0

*These data are found in page 278 in Reference 13. (See Section 6 for a complete list of references.)

Figure 3-1 is read as follows: An event with attribute A occurs a_1 times under conditions I, and a_2 times under conditions II. A mutually exclusive event with attribute B occurs b_1 times under condition I, and b_2 times under conditions II. Clearly, $N_A = a_1 + a_2$; $N_B = b_1 + b_2$, $N_1 = a_1 + b_1$; $N_2 = a_2 + b_2$; and $N = N_A + N_B = N_1 + N_2$.

To test a hypothesis developed in conjunction with a contingency table, use is made of the Chi-square statistical test. This statistical test is defined as follows for a k x m contingency table.

$$\chi^2 = \sum_{i=1}^{k} \sum_{j=1}^{m} \frac{(o_{ij} - e_{ij}^2)}{e_{ij}} ; \begin{cases} i=1, 2, \ldots , k \\ j=1, 2, \ldots , m \end{cases} \qquad (3.4)$$

in which o_{ij} = the observed frequency in the i,j^{th} cell in the contingency table, and e_{ij} = the corresponding expected or theoretical frequency in the same cell. Equation (3.4) may be simplified somewhat as follows:

$$\chi^2 = \sum_i \sum_j \frac{o_{ij}^2}{e_{ij}} - 2 \sum_i \sum_j o_{ij} + \sum_i \sum_j e_{ij}$$

$$\chi^2 = \sum_i \sum_j \frac{o_{ij}^2}{e_{ij}} - N , \qquad (3.5)$$

since

$$\sum_i \sum_j o_{ij} = \sum_i \sum_j e_{ij} = N .$$

Relating this to Figure 3-1, it is seen, for example, that

$$e_{12} = \frac{N_A N_2}{N} \qquad (3.6)$$

Likewise, the entire expression for the Chi-square statistic is from Figure 3-1:

$$\chi^2 = \frac{N}{N_A}\left(\frac{a_1^2}{N_1} + \frac{a_1^2}{N_2}\right) + \frac{N}{N_B}\left(\frac{b_1^2}{N_1} + \frac{b_2^2}{N_2}\right) - N \tag{3.7}$$

To translate the above into concrete AAES terms, consider the sample of RA-5C ejection data shown above. The following null hypothesis is to be tested at the 0.05 significance level:

> H_o: Introduction of the HS-1A ejection seat has not resulted in an increase in the number of neck injuries incurred upon ejection.

This null hypothesis can be stated another way. Thus,

> H_o: Neck injuries experienced upon ejection from the RA-5C aircraft are independent of ejection seat type (HS-1/HS-1A) used.

To test this hypothesis, a 2 x 2 contingency table was analyzed as shown below. A sample of 50 ejections was considered. The analysis, self-explanatory, follows from Table 3-5.

$$\chi^2_{c/_{d.f.=1}} = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{(f_{ij} - e_{ij})^2}{e_{ij}} \tag{3.8}$$

$$\chi^2_{c/_{d.f.=1}} = \frac{(5 - 1.4)^2}{1.4} + \frac{(5 - 8.6)^2}{8.6}$$

$$+ \frac{(2 - 5.6)^2}{5.6} + \frac{(38 - 34.4)^2}{34.4}$$

$$= 9.26 + 1.51 + 2.31 + .38 = 13.46$$

Theoretical value of the Chi-square statistic, denoted here by χ^2_T, has the numerical value

$$\chi^2_{T/0.05;d.f.=1} = 3.841.$$

Since $\chi_c^2 > \chi_{T/0.05}^2$, updating the seat has resulted in a statistically significant shift in neck injuries. Therefore, the null hypothesis is rejected at the 0.05 level of significance, and we conclude that, for the sample under investigation, updating the HS-1 to the HS-1A seat is highly correlated with increase in neck injuries. At this point, it is interesting to observe that all neck injuries in the sample discussed here occurred upon ejection from RA-5C aircraft using an HS-1A (updated) seat escape system. The Yates' correction factor was not applied to this Chi-square test. Application of Yates' correction factor yields $\chi_c^2 = 9.9772$, and the null hypothesis is still rejected.

A coefficient of contingency, C, can be computed for this contingency test.

$$C = \left(\frac{\chi^2}{\chi^2 + N}\right)^{1/2} \tag{3.9}$$

For the RA-5C sample studied,

$$C = \left(\frac{13.46}{13.46 + 50}\right)^{1/2} = 0.4605$$

The coefficient of contingency is analogous to a correlation coefficient in that it measures the strength of a relationship between the two variables analyzed in a contingency table.

A contingency table that would be of considerable interest is that of fatality/no fatality versus updated seat/no updated seat. This kind of statistical analysis is a relatively simple yet powerful tool in the general context of statistical analyses of the AAES ejection injury data.

Section 4.0

PROPOSED STATISTICAL ANALYSIS METHODOLOGY

4.0 Proposed Statistical Analysis Methodology

There are a wide variety of statistical analysis techniques that can be applied to the Automated Aircrew Escape Systems. These include the following:

- Data Analyses.
- Use of discrete parent probability density functions such as binomial, Poisson, and multinomial.
- Higher order contingency table analysis.
- Application of Bayesian statistical theory to refine a priori probability estimates.
- Application of non-parametric (distribution free) statistical techniques such as the run and trend tests.

Each of the above will be illustrated in the material that follows.

4.1 Data Analysis

An ejection related injury data classification scheme that would be definitive, amenable to statistical analysis, and in consonance with MORs classification is shown below. Here it is noted that numerical values are attached to the MORs classification. Advantages of employing such a scheme are the following: (1) ease of applying statistical/numerical techniques, and (2) ease of expanding the classification to obtain better subjective focus on a particular injury pattern, and (3) conformity and uniformity of injury classification. Illustrations of each of these are given below.

Injury Classification (Alpha)	Injury Category	Injury Classification (Numerical)	Expanded Injury Classification (Numerical)
G	No/Minimal	1	10-19
F	Minor	2	20-29
B	Major	3	30-39
A	Fatal	4	40-49
L & U	Lost and Unknown	5	50-59

An expanded injury classification scheme also could be used to order injuries in increasing order of severity. For example,

Expanded Injury Classification (Numerical)	Range of Injury Severity
10-19	No/minimal injury to minor injury
20-29	Minor to major (non-fatal) injury
30-39	Major (non-fatal) to fatal injury
40-49	Fatality (classify according to kind of fatality)
50-59	Lost and Unknown (classify according to probable cause)

The above numerical classification scheme would assist greatly in the major injury area, where injuries could vary from, say, a sprained arm to very severe contusions and lacerations. In addition, such a scheme would assist the statistical/numerical analyst in his application of various techniques to the injury data.

An example of injury classification by aircraft type for a set of actual ejections under combat conditions is shown in Table 23 in [1, pg. 28]. Detailed information about the various injuries is displayed in Table 24 in [1, pg. 28]. Injury classification was restricted to major, minor and no injury. These tables are reproduced here as Tables 4-1 and 4-2, respectively.

4.2 Discrete Probability Density Functions

Use of probability density functions to include the binomial parent density function now will be discussed in the context of AAES ejection related injury data. At the outset, three notes are made: (1) the data selected for analysis here are actual observations of A-6 ejection related operational injury data, (2) the data are classified in keeping with the scheme outlined in Section 4.1, above, and (3) the following percentages, based on actual MORs data, are assigned to the various injury categories:

TABLE 4-1*

INJURY CLASSIFICATION BY AIRCRAFT TYPE

	Major	Minor	None
A-4	10	19	14
A-6	4	2	2
A-7	3	4	1
F-4	7	11	7
F-8	5	8	3
RA-5C	1	1	2
TOTAL	30	45	29

Ejection Type	Number of Cases	Injury (Percent)		
		Major	Minor	None
Combat	104	29	43	28
Operational	972	19	44	37
USAF Combat	262	13	57	31

TABLE 4-2*

TYPE OF INJURY REPORTED

Injury	Number of Times Reported	Injury	Number of Times Reported
Contusions	40	Muscle tear	5
Strains	25	Open wounds	4
Lacerations (severe)	24	Abrasions	2
Small cuts	24	Acoustic trauma	1
Fractures (simple and compound)	22	Amputation of limb	1
General overall pain	13	Concussion	1
Burns (1st, 2nd, and 3rd degree)	10	Hematoma	1
Sprains (major)	10	Dislocation	1
Spinal compression	8	Disease (malaria)	1

*These are shown as Tables 23 and 24 in BioTechnology's Report [1, pg. 28].

1 - No/Minimal injury, 18.69%

2 - Minor injury, 35.52%

3 - Non-Fatal Major injury, 25.23%

4 - Fatality, 14.02%, and

5 - Lost and Unknown, 6.54%

The above information is summarized in Table 4-3.

TABLE 4-3

INJURY CLASSIFICATION WITH PERCENTAGES
(A-6 Operational Ejection Related Injuries During 1969-1975)

Injury Classification	Injuries	Percentage
1 - No injury/Minimal	20	18.69
2 - Minor injury	38	35.52
3 - Non-fatal major injury	27	25.23
4 - Fatality	15	14.02
5 - Other (Lost and Unknown)	7	6.54

4.2.1 The Binomial Probability Density Function

To consider an example of use of the binomial probability density function, assume that the probability of the occurrence of a known fatality, on an ejection-to-ejection basis, is 0.10. This assumption is not unreasonable for the following reasons: (1) over the time interval 1 January 1969 through 31 December 1975, the MORs catalogued 1,069 ejections from all aircraft, and (2) over the same time period, there were 117 known fatalities. Thus, the relative frequency of occurrence of a fatality is equal to 0.109. Now make the following definitions:

$$p_f = 0.1 = \text{probability of a fatality on each ejection.}$$

$$1 - p_f = 0.9 = \text{probability of other than a fatality on each ejection.}$$

Suppose now a sample consisting of 30 ejections is available for analysis. What is the probability that exactly 3 of the ejections will

result in fatalities? Using equation (4.3) and the above a priori probabilities from ejection-to-ejection, it is seen that:

$$f_1(3) = \binom{30}{3} (0.1)^3 (0.9)^{30-3} = 0.236087$$

Using this information, together with the additional assumption that future ejections occur under conditions similar to those which have occurred in the past, how many fatalities can be expected to occur during the course of the next 50 ejections? If the assumption is made that f(x) for this problem is the same as f(3) for the preceding problem, then

$$f_2(x) = 0.236087 = \binom{50}{x} \left(\frac{1}{10}\right)^x \left(\frac{9}{10}\right)^{50-x} \tag{4.1}$$

Equation (4.1) can be written as follows:

$$\Delta(x) = 45.8086 - \left(\frac{50!}{x!\,(50 - x)!}\right) \left(\frac{1}{9}\right)^x , \tag{4.2}$$

where it is desired to find an x such that $\Delta(x)$ is minimized. From equation (4.2), Table 4-4 is derived:

TABLE 4-4

VALUES OF $\Delta(x)$ VERSUS x

x	$\Delta(x)$
1	40.253
2	25.129
3	18.922
4	10.707
5	9.927
6	15.907
7	24.925

Not surprisingly, we should expect 5 fatalities during the next 50 ejections, where the ejections are conducted under conditions comparable to those under

which ejections have been taking place during the recent past. Note that $x = 5$ is the value of x which minimizes $\Delta(x)$. A graph of $\Delta(x)$ versus x is shown in Figure 4-1.

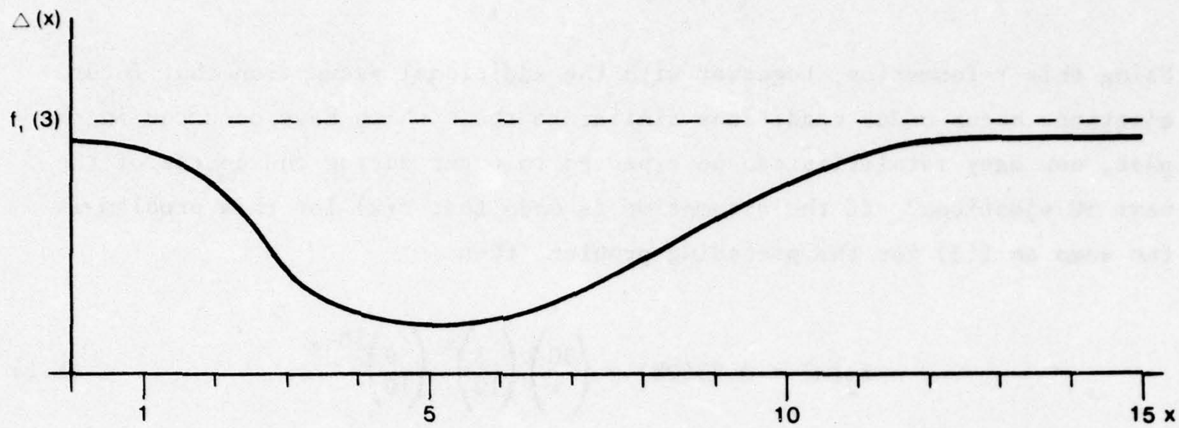

FIGURE 4-1. GRAPH OF $\Delta(x)$ VERSUS x

Analyses similar to the above could be performed on the injury categories entitled 1 - no injury, 2 - minor injury, and 3 - non-fatal major injury.

As another example of application of the binomial probability density function, consider the dichotomous injury data presented earlier, namely 1 = neck injury and 0 = no neck injury. Underlying this binary injury designation is the idea of a probability density function. Thus, there is a probability measure attached to each designation. For example, it seems clear to say that if an ejectee will have a neck injury, with a given probability, then he will have no neck injury with a probability equal to one minus the probability that he will have a neck injury. In symbols, let

p = probability of neck injury, then

$q = (1 - p)$ = probability of no neck injury.

Immediately these relate to the binomial probability density function defined as follows:

$$f(x) = \binom{n}{x} p^x q^{n-x}$$

where $f(x)$ = binomial probability density function

$\binom{n}{x}$ = binomial coefficient = $\dfrac{n!}{x!\,(n-x)!}$

p = probability of occurrence of an event with a given attribute, A,

q = (1 - p) = probability of non-occurrence of the event with a given attribute, A,

x = number of occurrences observed

n = number of experiments performed during the course of which x occurrences were observed

To relate this binomial probability density function to the example discussed in Section 3.1, let the a priori probability of occurrence of a neck injury be defined as the relative frequency of occurrence of a neck injury. Then,

$$p = 3/17$$

$$q = 14/17 = 1 - p$$

Hence,

$$f(x) = \binom{n}{x} \left(\frac{3}{17}\right)^x \left(\frac{14}{17}\right)^{n-x}$$

To compute some numbers, let x = 1, 2, 3, and 17. Corresponding values of f(x) are:

$$f(1) = \binom{17}{1} \left(\frac{3}{17}\right)^1 \left(\frac{14}{17}\right)^{17-1}$$

$$f(1) = 0.13427$$

$$f(2) = 0.23018$$

$$f(3) = 0.24662$$

$$f(17) = 3.6265 \times 10^{-21}$$

Again, it should be stated that these numbers were generated using the relative frequency as the a priori probability $p = 3/17$ of a neck injury.

These a priori probability measures can doubtless be refined by Bayesian techniques, if sufficient injury/no injury data are available. This possibility will be discussed in more depth presently.

To summarize, it is believed imperative that some parent probability density function be associated with a binary injury/no injury data analysis procedure.

4.2.2 The Poisson Probability Density Function

As mentioned previously, the binomial probability density function is defined by the equation

$$f_b\ (n;\ x,\ p) = \binom{n}{x} p^x\ (1 - p)^{n-x}\ ; \tag{4.3}$$

in which

n = number of occurrences of an event
x = number of successes in a given experiment
p = probability of success on a given trial within the experiment.

When n is large, direct calculation of probabilities using equation (4.3) involves an enormous amount of calculation. As an example, suppose it is known a priori that the probability of an individual getting frostbitten on a cold winter day while attending a sporting event is 0.001. What is the probability that 27 persons in a crowd of 30,000 will get frostbitten? Direct substitution into equation (4.3) yields:

$$f_b(30{,}000;\ 27,\ 0.001) = \binom{30{,}000}{27} (0.001)^{27}\ (1 - 0.001)^{29{,}973} \tag{4.4}$$

Considerable effort would be required to solve equation (4.4).

If the observation is made that n is large and p is small such that np = constant as $n \to \infty$ and $p \to 0$ then the Poisson probability function can be useful. Let $np = \lambda$, a constant as $n \to \infty$ and $p \to 0$. Then write equation (4.3) as follows:

$$f_b\ (n;\ x,\ p) = \frac{(n)\ (n-1)\ (n-2)\ \ldots\ (n-x+1)}{x!} \left(\frac{\lambda}{n}\right) \left(1 - \frac{\lambda}{n}\right)^{n-x} \qquad (4.5)$$

Write

$$\left(1 - \frac{\lambda}{n}\right)^{n-x} = \left[\left(1 - \frac{\lambda}{n}\right)^{-n/\lambda}\right]^{-\lambda} \left(1 - \frac{\lambda}{n}\right)^{-x}$$

then substitute into equation (4.5) to get:

$$f_b\ (n;\ x,\ p) = \frac{(1)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{x-1}{n}\right)}{x!}\ \lambda^x$$

$$\cdot \left[\left(1 - \frac{\lambda}{n}\right)^{-n/\lambda}\right]^{-\lambda} \left(1 - \frac{\lambda}{n}\right)^{-x}$$

Take the limit as $n \to \infty$ and λ = constant to get:

$$f_p\ (x;\ \lambda) = \left(\frac{\lambda^x}{x!}\right) \left(e^{-\lambda}\right) \qquad (4.6)$$

which is the Poisson probability density function.

The preceding problem can now be worked with considerable ease. Thus,

$$np = (30{,}000)\ (0.001) = 30$$
$$x = 27.$$

Substitute into equation (4.6) to get:

$$f_p\,(27;\,30) = \frac{30^{27}}{27!}\,e^{-30}$$

$$f_p\,(27;\,30) = 0.06553$$

Statistical examples of use of the Poisson probability density function are sometimes called rare events and occur in widely different fields. For example, the number of individuals born blind per year in a large city; the number of organisms of a given size S on a glass slide that escape death by X-rays after having been exposed for t-seconds; the number of times in a year that the volume of shares traded on the New York Exchange will exceed M-million; number of peak loads in telephone traffic during a given period of time, etc.

To apply the Poisson probability density function to the Aircrew Automated Escape System, suppose it is known that 5000 propulsion actuated escape devices have been manufactured. What is the probability of there being 2 devices among the 5000 that will fail to fire, given the probability of an individual failing to fire as 0.005? Here, $n = 5{,}000$, $p = 0.005$, $\lambda = np = 25$, $x = 2$. Substitute into equation (4.6) to get:

$$f_p\,(2;\,25) = \frac{25^2\,e^{-25}}{2!} = 0.0000000043.$$

There is thus a very small chance that exactly two devices out of the 5,000 will fail to fire.

A question sometimes asked is the following: What is the probability of there being more than two devices that will fail to fire? To answer this question, examine the cumulative Poisson distribution function. Thus,

$$\text{Prob }(x = 0 \text{ or } 1, \text{ or } 2) = \sum_{x=0}^{2} f(x; \lambda)$$

From equation (4.6),

$$\text{Prob }(x = 0 \text{ or } 1 \text{ or } 2) = e^{-25}\left[1 + 25 + \frac{(25)^2}{2}\right]$$

$$= 0.0000000047$$

Hence,

$$P\ (x > 2) = 1 - \text{Prob }(x = 0, \text{ or } 1, \text{ or } 2),$$

and

$$P\ (x > 2) = 0.9999999953.$$

Thus, it is seen that, under the conditions assumed, there is a very high probability that more than two devices will fail to fire. If $p = 0.0001$ above, then $P\ (x > 2) = 0.01439$. This points out the necessity for high reliability in each hardware component.

4.2.3 The Multinomial Probability Density Function

The binomial experiment becomes a multinomial experiment if each trial can have more than two possible outcomes. In a general case, if a given trial can result in any one of k possible outcomes E_1, E_2, ... , E_k, with probabilities p_1, p_2, ... , p_k then the probability density function of the random variables x_1, x_2, ... , x_k, representing the number of occurrences for the events E_1, E_2, ... , E_k, in n independent trials is defined by the equation:

$$f_m(\vec{x}; \vec{p}; n) = \binom{n}{x_1, x_2, \ldots, x_k} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} \tag{4.7}$$

Equation (4.7) may be written compactly as follows:

$$f_m(\vec{x}; \vec{p}; n) = n! \prod_{i=1}^{k} \frac{p_i^{x_i}}{(x_i)!} \tag{4.8}$$

where

$$\vec{x} = x_1, x_2, \ldots, x_k; \sum_{i=1}^{k} x_i = n,$$

and

$$\vec{p} = p_1, p_2, \ldots, p_k; \sum_{i=1}^{k} p_i = 1,$$

and

$$\binom{n}{\vec{x}} = \frac{n!}{x_1!\, x_2! \ldots x_k!}; \tag{4.9}$$

A partial graph of a trinomial probability density function is shown in Figure 4-2.

To relate the multinomial probability density function to the AAES ejection related injury problem, refer to data in Table 4-3 which contains the injury history of ejectees from the A-6 aircraft over the time span 1 January 1969 through 31 December 1975. Using the relative frequency of occurrence of an injury as its individual probability of occurrence, what is the probability of obtaining exactly the injury pattern shown in Table 4-3? Under the assumptions made, $P_1 = 0.1869$, $P_2 = 0.3552$, $P_3 = 0.2523$, $P_4 = 0.1402$, and $P_5 = 0.0654$. Also, $x_1 = 20$, $x_2 = 38$, $x_3 = 27$, $x_4 = 15$ and $x_5 = 7$.

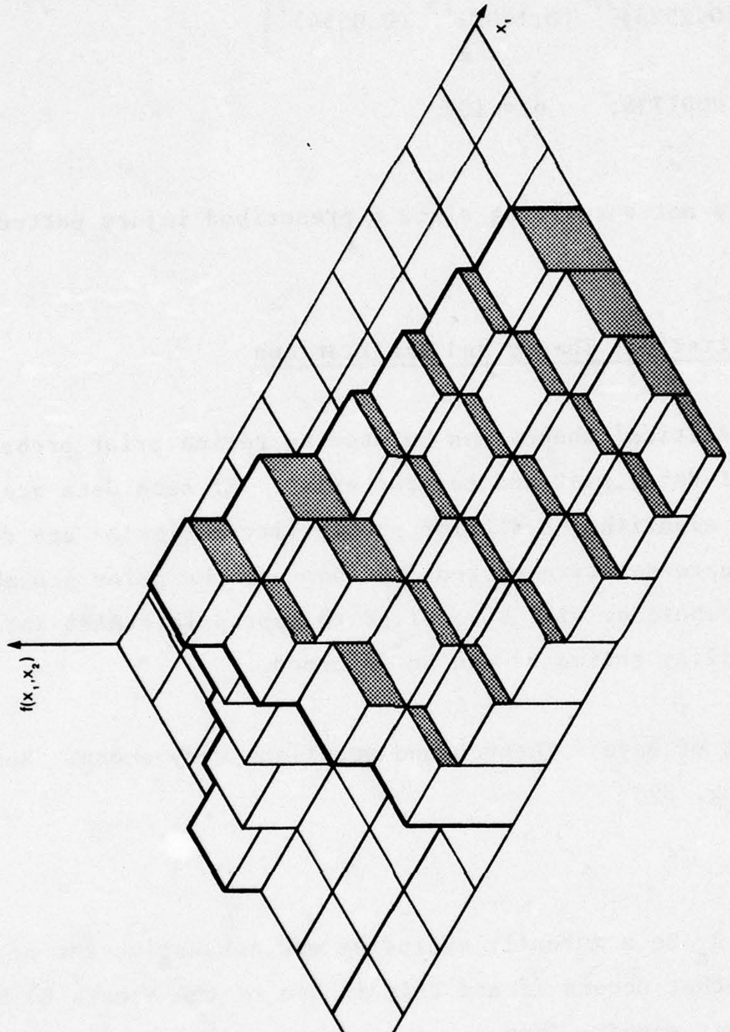

FIGURE 4-2. GRAPH OF A TRINOMIAL PROBABILITY DENSITY FUNCTION

Substitute these numbers into equation (4.8) to get:

$$f_m(\vec{x}; \vec{p}; n) = \frac{(107)!}{20!\ 38!\ 27!\ 15!\ 7!}\Big[(0.1869)^{20}\ (0.3552)^{38}$$

$$\cdot (0.2523)^{27}\ (0.1402)^{15}\ (0.0654)^{7}\Big]$$

$$f_m(\vec{x}; \vec{p}; n) = 0.0001738; \qquad n = 107$$

This low probability is not surprising since a prescribed injury pattern was given.

4.3 Bayesian Statistical Theory and Applications

Bayesian statistical theory can be used to refine prior probability estimates if sufficient data, past and current exist. If such data are subjected to the Bayesian algorithm, posterior probability estimates are calculated which are more representative of reality than are the prior probability estimates. Bayesian techniques will be applied to appropriate AAES data sets so that refined probability *estimates can be inferred*.

The statement of Bayes' Theorem and proof are very short. Both are given below. [2, pg. 598]

Theorem:

Let $B_1, B_2, \ldots, B_n$ be a mutually exclusive and exhaustive set of events. Let E be another event that occurs if and only if one of the events B_j has occurred. Let B_i be that event. Then,

$$P(B_i \mid E) = \frac{P(B_i)\, P(E \mid B_i)}{\sum_{j=1}^{n} P(B_j)\, P(E \mid B_j)} \tag{4.10}$$

Proof:

Use the general multiplication rule in both its forms, thus:

$$P\,(B_i\,E) = P\,(B_i)\;P\,(E|\,B_i) \tag{4.11}$$

$$P\,(B_i\,E) = P\,(E)\;P\,(B_i|\,E) \tag{4.12}$$

Due to the fact that the character of B_j is mutually exclusive and exhaustive,

$$P\,(E) = \sum_{j=1}^{n} P\,(B_j)\;P\,(E|\,B_j) \tag{4.13}$$

Write equation (4.12) thus

$$P\,(B_i|\,E) = \frac{P\,(B_i\,E)}{P\,(E)} \tag{4.14}$$

Now substitute equations (4.11) and (4.13) into equation (4.14).

$$P\,(B_i|\,E) = \frac{P\,(B_i)\;P\,(E|\,B_i)}{\sum_{j=1}^{n} P\,(B_j)\;P\,(E|\,B_j)},$$

which is equation (4.10) and proves the theorem.

To use Bayesian Statistical Theory in an analysis of the AAES ejection/fatality problem consider the following situation: A study was performed on the A-6 ejection/fatality problem. A sample of operational ejections from the A-6 aircraft was studied each year for the past six years. The fraction of fatalities occurring during each year, here defined as the ratio of fatalities to ejections, or fatality rate, is shown as column 4 in Table 4-5. The actual number of fatalities year-by-year is listed in column 3. The relative frequency of occurrence of fatalities, here designated as the a priori

probability of occurrence of fatalities, is displayed in column 5. Three fatalities have occurred during the course of 16 ejections in the current year. Use this information together with Bayesian Statistical Theory, to develop improved numbers for the a priori probability of occurrence of fatalities.

TABLE 4-5

ANALYSIS OF A-6 EJECTION RELATED FATALITIES

YEAR	EJECTIONS	FATALITIES	FRACTION OF FATALITIES (Col.3 ÷ Col.2)	RELATIVE FREQUENCY (PRIOR PROBABILITY) (Col.3 ÷ Σ Col.3)
(1)	(2)	(3)	(4)	(5)
1969	14	5	0.3571	0.2778
1970	17	3	0.1765	0.1667
1971	22	0	0.0000	0.0000
1972	18	5	0.2778	0.2777
1973	10	2	0.2000	0.1111
1974	8	3	0.3750	0.1667
TOTAL:	89	18	0.2022	1.0000

Now, make use of the numbers in Table 4-5, together with Bayesian Statistical Theory to construct Table 4-6. The purpose of Bayesian statistical analysis is to use current information to refine previously derived results.

If suitable data exist, Bayesian statistical theory appears useful in the analysis of dichotomous ejection/injury problem area of this AAES study. As mentioned, it is a very useful algorithm for refining a priori probability estimates of the occurrence of various events.

4.4 Higher Order Contingency Table Data Analysis

Some information about contingency tables was developed in Section 3.2. Here, an example of a higher order contingency table, and analysis which accompanies it, will be presented. Data for the work that follows were taken from [1, pg. 28]. These data are based on actual combat ejection observations.

TABLE 4-6

APPLICATION OF BAYESIAN STATISTICAL THEORY TO THE A-6 EJECTION/FATALITY PROBLEM

Fraction of Fatalities (p) (Found in Table 4-5)	Prior Probability P(p) (Found in Table 4-5)	Conditional Probability* =P (r=3, | n=16, p) *(Found in Tables of the Binomial Density Function)	Joint Probability $P(p) \cdot P(r=3 \mid n=16,p)$ (Col. 2 x Col. 3)	Posterior Probability $\frac{P(p) \cdot P(r=3 \mid n=16, p)}{\Sigma P(p) \cdot P(r=3 \mid n=16,P)}$ (Col. 4 ÷ Σ Col. 4)
(1)	(2)	(3)	(4)	(5)
0.3571	0.2778	0.08174	0.02271	0.15079
0.1765	0.1667	0.24663	0.04111	0.27302
0.0000	0.0000	0.00000	0.00000	0.00000
0.2778	0.2777	0.17456	0.04847	0.32190
0.2000	0.1111	0.24629	0.02736	0.18171
0.3750	0.1667	0.06557	0.01093	0.07258
TOTALS:			Σ = 0.15059	Σ = 1.00000

Marginal Probability for this sample of Ejections.

Ejection/injury data for six high performance aircraft are presented in Table 4-7. Here, it should be noted that two injury categories are studied: (1) major injuries, and (2) minor injuries.

TABLE 4-7

COMBAT EJECTION INJURIES VERSUS AIRCRAFT TYPE

Injury ↓ \ Aircraft/Seat →	A-4/ ESCAPAC	A-6/ Mk GRU-7	A-7/ ESCAPAC IC-3	F-4/ Mk H-7	F-8/ Mk F-7	RA-5C/ HS-1/ HS-1A	TOTAL:
Major	10	4	3	7	5	1	30
Minor	19	2	4	11	8	1	45
TOTAL:	29	6	7	18	13	2	75

From the data in Table 4-7, the following null hypothesis is proposed:

H_o: Ejection injury is independent of aircraft/seat type from which the ejection was made.

Consider now the universe of 75 ejection related injuries experienced in combat by aircrewman upon ejection from the six different aircraft shown in Table 4-7. The probability that a particular injury, from this universe of injuries, occurred upon ejection from an A-4 aircraft is $P(A_4)$. Numerically, from Table 4-7, $P(A_4) = 29/75$.

Similarly,

$$P(A_6) = \frac{6}{75} \; ; \; P(A_7) = \frac{7}{75} \; ; \; P(F_4) = \frac{18}{75} \; ; \; P(F_8) = \frac{13}{75} \; ;$$

and

$$P(A_5) = \frac{2}{75} .$$

Likewise from Table 4-7,

$$P(M) = \frac{30}{75} = \text{Probability of a major injury}$$

$$P(m) = \frac{45}{75} = \text{Probability of a minor injury}$$

Under the hypothesis H_o, that is, ejection injury is independent of aircraft/seat type,

$P(A_4 \cap M) = P(A_4)\ P(M)$ = Probability acquiring a major injury upon ejection from an A-4 aircraft.

Thus,

$$P(A_4 \cap M) = \left(\frac{29}{75}\right)\left(\frac{30}{75}\right) = 0.155$$

Similarly,

$$P(A_6 \cap M) = P(A_6)P(M) = \left(\frac{6}{75}\right)\left(\frac{30}{75}\right) = 0.032$$

$$P(A_7 \cap M) = \left(\frac{7}{75}\right)\left(\frac{30}{75}\right) = 0.037$$

$$P(F_4 \cap M) = \left(\frac{18}{75}\right)\left(\frac{30}{75}\right) = 0.096$$

$$P(F_8 \cap M) = \left(\frac{13}{75}\right)\left(\frac{30}{75}\right) = 0.069$$

$$P(A_5 \cap M) = \left(\frac{2}{75}\right)\left(\frac{30}{75}\right) = 0.011$$

Analogously, under H_o,

$P(A_4 \cap m) = P(A_4)\ P(m)$ = Probability of acquiring a minor injury upon ejection from an A-4 aircraft.

$$P(A_4 \cap m) = \left(\frac{29}{75}\right)\left(\frac{45}{75}\right) = 0.232$$

$$P(A_6 \cap m) = \left(\frac{6}{75}\right)\left(\frac{45}{75}\right) = 0.048$$

$$P(A_7 \cap m) = \left(\frac{7}{75}\right)\left(\frac{45}{75}\right) = 0.056$$

$$P(F_4 \cap m) = \left(\frac{18}{75}\right)\left(\frac{45}{75}\right) = 0.144$$

$$P(F_8 \cap m) = \left(\frac{13}{75}\right)\left(\frac{45}{75}\right) = 0.104$$

$$P(A_5 \cap m) = \left(\frac{2}{75}\right)\left(\frac{45}{75}\right) = 0.016$$

To calculate theoretical, that is expected, frequency of injury, multiply each probability above by the total number of injuries. Thus,

$E_m(A_4) = 75\ P(A_4 \cap M)$ = Expected number of major injuries upon ejection from an A-4 aircraft.

$$E_M(A_4) = 75\ P(A_4 \cap M) = (75)\ (0.155) = 11.6 = 12$$

$$E_M(A_6) = (75)\ (0.032) = 2.4 = 2$$

$$E_M(A_7) = (75)\ (0.037) = 2.8 = 3$$

$$E_M(F_4) = (75)\ (0.096) = 7.2 = 7$$

$$E_M(F_8) = (75)\ (0.069) = 5.2 = 5$$

$$E_M(A_5) = (75)\ (0.011) = 0.8 = 1$$

Likewise, the expected number of minor injuries is found as follows:

$$E_m(A_4) = (75)\ (0.232) = 17.4 = 17$$

$$E_m(A_6) = (75)\ (0.048) = 3.6 = 4$$

$$E_m(A_7) = (75)\ (0.056) = 4.2 = 4$$

$$E_m(F_4) = (75)\ (0.144) = 10.8 = 11$$

$$E_m(F_8) = (75)\ (0.104) = 7.8 = 8$$

$$E_m(A_5) = (75)\ (0.016) = 1.2 = 1$$

From these data, based on actual combat ejection observations, Table 4-8 is constructed.

TABLE 4-8

COMPARISON OF OBSERVED FREQUENCY VERSUS THEORETICAL FREQUENCY OF COMBAT EJECTION INJURY OCCURRENCE (Theoretical Frequency is Shown in Parenthesis)

Aircraft/Seat → / Injury ↓	A-4/ ESCAPAC	A-6/ Mk GRU-7	A-7/ ESCAPAC IC-3	F-4/ Mk H-7	F-8/ Mk F-7	RA-5C/ HS-1/ HS-1A	TOTAL:
Major	10(12)	4(2)	3(3)	7(7)	5(5)	1(1)	30(30)
Minor	19(17)	2(4)	4(4)	11(11)	8(8)	1(1)	45(45)
TOTAL:	29(29)	6(6)	7(7)	18(18)	13(13)	2(2)	75(75)

The proposed hypothesis H_o is tested with the Chi-square statistic. Thus,

$$\chi^2_{c/d.f.=5} = \sum_{c=1}^{6} \sum_{r=1}^{2} \frac{(o_{r,c} - e_{r,c})^2}{e_{r,c}} \qquad (4.15)$$

ν = degrees of freedom, where $\nu = (r - 1) \cdot (c - 1)$, here r = 2 c = 6, hence ν = 5 degrees of freedom. Note: r = rows, c = columns.

$o_{r,c}$ = the observed frequency at the "point" r, c, in Table 4-8.

$e_{r,c}$ = the expected frequency at the "point" r, c.

From Table 4-8, the Chi-square statistic is easy to compute. Thus,

$$\chi^2_{c/d.f.=5} = \frac{(10-12)^2}{12} + \frac{(4-2)^2}{2} + \frac{(3-3)^2}{3} + \frac{(7-7)^2}{7}$$

$$+ \frac{(5-5)^2}{5} + \frac{(1-1)^2}{1} + \frac{(19-17)^2}{17} + \frac{(2-4)^2}{4}$$

$$+ \frac{(4-4)^2}{4} + \frac{(11-11)^2}{11} + \frac{(8-8)^2}{8} + \frac{(1-1)^2}{1}$$

$$\chi^2_{c/d.f.=5} = 3.569$$

From Tables of the Chi-square statistic for ν = 5 degrees of freedom at $\alpha = 0.05$,

$$\chi^2_{0.05/d.f.=5} = 11.07$$

Since $\chi^2_c < \chi^2_{0.05}$ for 5 degrees of freedom, accept the null hypothesis at the $\alpha = 0.05$ level of significance and conclude that ejection injury and the aircraft/seat type from which the ejection was made are independent. This Chi-square test was performed without applying the Yates' correction factor. Application of Yates' correction factor yields

$$\chi^2_{c/d.f.=5} = 3.027,$$

and the null hypothesis is still accepted.

Contingency table data analysis has been demonstrated, on the Ballistic Spreader Gun problem, to be a very powerful tool in the analysis of ejection seat data.

In the interest of some generality, an extension of the contingency tables presented thus far now will be given. The generalization will be to a contingency table having m-rows and n-columns. It will be known as an m x n contingency table. An example is displayed in Table 4-9.

Hypothesis testing, using the Chi-square statistic in conjunction with the m x n contingency table, is an extension of equation (3.7) which is used to perform hypothesis testing with a 2 x 2 contingency table. The equation follows.

$$\chi^2 /_{d.f.=(m-1)(n-1)} = \frac{N}{N_{1,n}} \sum_{i=1}^{n} \frac{a_{1,i}^2}{M_{m,i}} + \frac{N}{N_{2,n}} \sum_{i=1}^{n} \frac{a_{2,i}^2}{M_{m,i}} + \dots + \frac{N}{N_{m,n}} \sum_{i=1}^{n} \frac{a_{m,i}^2}{M_{m,i}} - N \qquad (4.16)$$

In compact notation, equation (4.16) is written:

$$\chi^2 /_{d.f.=(m-1)(n-1)} = \sum_{j=1}^{m} \left[\frac{N}{N_{j,n}} \sum_{i=1}^{n} \frac{a_{j,i}^2}{M_{m,i}} \right] - N \qquad (4.17)$$

The coefficient of contingency can be found by using equation (3.9).

TABLE 4-9

A GENERALIZED m x n CONTINGENCY TABLE

Cause → / Effect ↓	$A_{m,1}$	$A_{m,2}$	...	$A_{m,n}$	TOTALS:
$A_{1,n}$	$a_{1,1}$	$a_{1,2}$	...	$a_{1,n}$	$N_{n,1} = \sum_{i=1}^{n} a_{1,i}$
$A_{2,n}$	$a_{2,1}$	$a_{2,2}$	...	$a_{2,n}$	$N_{n,2} = \sum_{i=1}^{n} a_{2,i}$
⋮	⋮	⋮	...	⋮	⋮
$A_{m,n}$	$a_{m,1}$	$a_{m,2}$	...	$a_{m,n}$	$N_{n,m} = \sum_{i=1}^{n} a_{m,i}$
TOTAL:	$M_{m,1} = \sum_{j=1}^{m} a_{j,1}$	$M_{m,2} = \sum_{j=1}^{m} a_{j,2}$	...	$M_{m,n} = \sum_{j=1}^{m} a_{j,n}$	$N = \sum_{j=1}^{m} N_{n,j} = \sum_{i=1}^{n} M_{m,i}$

It may happen that small values are given for both m and n. In this case, equation (4.15), and more generally equation (4.17), represent only approximations to the Chi-square statistic. Thus, a correction, due to Yates' should be applied. As was suggested by Yates', [7, pg. 230; pg. 259], the approximation to χ^2 is improved by replacing one cell frequency, say $a_{i,j}$ by $a_{i,j} \pm 1/2$, and adjusting the other elements in Table 4-9 so as to keep the marginal totals unaltered. Writing an alternate form of equation (4.17) as follows (here let $a_{i,j}$ = observed frequency in the i,j^{th} cell):

$$\chi^2/_{d.f.=(m-1)(n-1)} = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\left(a_{i,j} - e_{i,j}\right)^2}{e_{i,j}} \tag{4.18}$$

Application of the Yates' correction to equation (4.18) yields the following:

$$\chi^2/_{d.f.=(m-1)(n-1)} = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\left(|a_{i,j} - e_{i,j}| - 0.5\right)^2}{e_{i,j}} \tag{4.19}$$

where from Table 4-9,

$$e_{i,j} = \frac{N_{i,n} \ M_{m,j}}{N}, \tag{4.20}$$

and $e_{i,j}$ = expected frequency in the i,j^{th} cell,

$N_{i,n}$ = total of observed frequencies in the i^{th} row

$M_{m,j}$ = total of observed frequencies in the j^{th} column.

4.5 Non-parametric (Distribution Free) Statistical Tests

Several methods have been developed recently which make it possible to judge the randomness of observed data on the basis of the order in which the observations are obtained. Thus a test can be performed to determine, within probabilistic limits, whether the data contain patterns that look suspiciously non-random. It is interesting to note that this test can be applied after the data are collected. The technique is based on the theory of runs. This is a non-parametric statistical technique. Another non-parametric technique, to be presented shortly, is the trend analysis of data.

4.5.1 The Run Test

A run is a succession of identical symbols which is followed and preceded by different symbols or no symbols at all. To illustrate, let n = non-defective pieces and d = defective pieces produced by a given machine. Suppose the pieces manufactured by the machine form a run as follows:

(n, n, n, n, n,)(d, d, d, d,)(n, n, n, n, n, n, n, n, n, n,)(d, d,)
(n, n,)(d, d, d, d,)(n,)(d, d,)(n, n,)

Here, there are 5 non-defectives, followed by 4 defectives, etc...finally 2 defectives followed by 2 non-defectives. Each set of symbols in parenthesis represents a run. In this example, there are $u = 9$ runs, $n_1 = 20$ non-defectives and $n_2 = 12$ defectives.

The total number of runs appearing in an arrangement of this kind is often a good indication of a possible lack of randomness. If there are too few runs, a definite grouping or clustering might be suspected, perhaps even a trend. If too many runs are observed, some sort of repeated alternating pattern might be suspected.

The probability density function for the distribution of u-runs is [5, pg. 353]:

$$f(u) = \frac{2\binom{n_1 - 1}{k - 1}\binom{n_2 - 1}{k - 1}}{\binom{n_1 + n_2}{n_1}}; \quad u = 2k; \tag{4.21}$$

k is a positive integer and

$$f(u) = \frac{\binom{n_1 - 1}{k}\binom{n_2 - 1}{k - 1} + \binom{n_1 - 1}{k - 1}\binom{n_2 - 1}{k}}{\binom{n_1 + n_2}{n_1}}; \quad u = 2k+1 \tag{4.22}$$

In equations (4.21) and (4.22),

u = the number of runs,

n_1 = the number of observations having a given attribute, 1, say,

n_2 = the total number of observations less the number of observations, n_1, having a given attribute, 1, say.

In this example, it should be reiterated that a dichotomous situation is under observation. That is, an event either has a given attribute, or it does not have that attribute.

To consider a concrete example applicable to the AAES, a sample of fifty A-4 operational ejections was observed, and injuries classified according to the injury classification scheme shown in Table 4-3. The following sequence of observations was made.

TABLE 4-10

A-4 EJECTION INJURY DATA

1, 1, 5, 4, 2, 1, 3, 4, 2, 3, 1, 3, 4, 1, 1, 4, 3,
2, 1, 2, 4, 2, 2, 1, 2, 2, 1, 1, 3, 1, 2, 1, 1, 2,
2, 1, 2, 1, 2, 1, 3, 2, 1, 4, 1, 1, 1, 1, 1, 1.

It was decided to partition the injuries into two sets: (1) classifications 1, and 2, that is no injuries and minor injuries, and (2) classifications 3, 4, and 5 -- major injuries, fatalities and other. This is now a dichotomous situation. Let n represent the first set, and i represent the second set. The preceding run can now be written in dichotomous notation as follows:

n, n, i, i, n, n, i, i, n, i, n, i, i, n, n, i, i,
n, n, n, i, n, n, n, n, n, n, n, i, n, n, n, n, n,
n, n, n, n, n, n, i, n, n, i, n, n, n, n, n, n.

From these data, the following number are derived:

n_1 = non-major injured = 37

n_2 = major injured, including fatalities = 13

u = number of runs = 19

Expressions for the mean and variance of u when n_1 and n_2 both exceed 10 are defined as follows [5, pg. 354]:

$$E(u) = \frac{2\, n_1 n_2}{n_1 + n_2} + 1 \,, \tag{4.23}$$

and

$$Var(u) = \frac{2\, n_1\, n_2\, (2\, n_1\, n_2 - n_1 - n_2)}{(n_1 + n_2)^2\, (n_1 + n_2 - 1)} \tag{4.24}$$

Substituting from above for n_1 and n_2:

$$E(u) = \frac{(2)\ (37)\ (13)}{37 + 13} + 1 = 20.24$$

$$Var(u) = \frac{(2)\ (37)\ (13)\ [(2)\ (37)\ (13) - 37 - 13]}{(37 + 13)^2\ (37 + 13 - 1)}$$

$$Var(u) = 7.161992$$

$$\sqrt{Var(u)} = 2.67619$$

The null hypothesis being tested is the following:

H_o: The dichotomous A-4 injury data shown in Table 4-10 is comprised of two random samples of n's and i's which have the same parent population density function.

This hypothesis is to be tested at the $\alpha = 0.05$ level of significance with the statistic

$$z_c = \frac{u - E(u)}{\sqrt{Var(u)}} \tag{4.25}$$

under the assumption that n_1 and n_2 are sufficiently large (greater than 10) so that the sampling distribution of u can be approximated with a normal density function. Substituting numbers generated into equation (4.25) yields:

$$z_c = \frac{19.0 - 20.24}{2.67619} = -\ 0.46335$$

Since $z_{\alpha/2} = z_{0.025} = 1.96$, it is clear that $-\ 1.96 < -\ 0.46335 < +\ 1.96$, thus the null hypothesis is accepted at the $\alpha = 0.05$ level of significance, and we conclude that both samples are extracted from the same parent population density function.

It would be interesting to compute f(u) from equation (4.21). Thus, using the numbers developed above, that is, $u = 19 = 2k + 1$, hence $k = 9$, $n_1 = 37$, and $n_2 = 13$. Thus the probability of observing exactly 19 runs is

$$f(19) = \frac{\binom{36}{9}\binom{12}{8} + \binom{36}{8}\binom{12}{9}}{\binom{50}{37}}$$

$$f(19) = 0.150082$$

The familiar concept of "level-of-significance" is illustrated, for this particular problem, by Figure 4-3. Since n_1 and n_2 are both greater than 10, the sampling distribution of u is assumed to be normal. Recall that "level-of-significance" is the probability of a Type I error. It is the probability of rejecting an hypothesis when in fact the hypothesis is true.

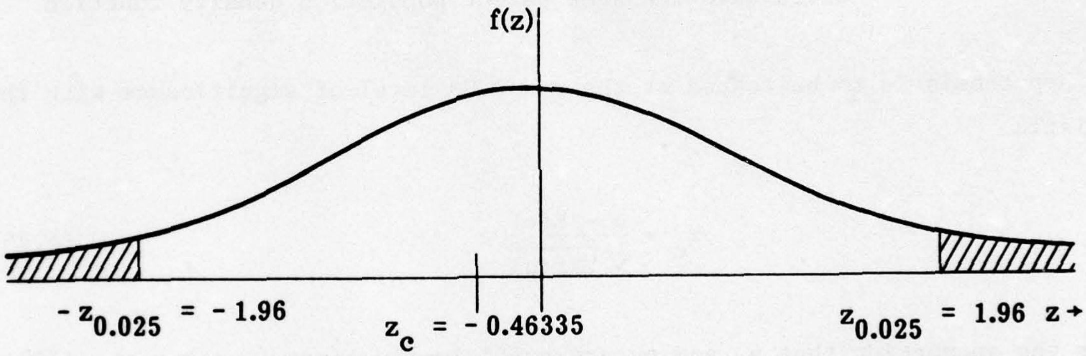

FIGURE 4-3. ILLUSTRATION OF LEVEL OF SIGNIFICANCE, WHERE $\alpha = 0.05$ FOR A TWO-TAIL TEST

In Figure 4-3,

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \tag{4.26}$$

where z is standardized by equation (4.25). Clearly,

$$\int_{-\infty}^{1.96} f(z)\,dz = \int_{-\infty}^{0} f(z)\,dz + \int_{0}^{1.96} f(z)\,dz$$

$$= 0.5000 + 0.4750 = 0.9750$$

Hence

$$\int_{1.96}^{\infty} f(z)\,dz = \int_{-\infty}^{-1.96} f(z)\,dz = 1.0 - 0.975$$

Thus,

$$\int_{-\infty}^{-1.96} f(z)\,dz = 0.025.$$

For a two-tail test:

$$\alpha = \int_{-\infty}^{-1.96} f(z)\,dz + \int_{1.96}^{\infty} f(z)\,dz = 0.025 + 0.025$$

Thus, $\alpha = 0.05$. Now, since $z_c = -\ 0.46335$, clearly

$$-\ 1.96 < z_c < +\ 1.96,$$

that is, z_c falls in the acceptance region. Hence at the $\alpha = 0.05$ level of significance, the null hypothesis is accepted.

4.5.2 Tests on Runs Above and Below the Median

The theory of runs above and below the median also may be used to test the hypothesis that observations have been drawn at random from a single population. To perform this experiment, find the median of the sample and denote observations above the median by an a, and those below the median by a b. If the number of runs of a's and b's is larger or smaller than might be expected by chance, reject the hypothesis that the observations have been drawn at random from a single population.

From the data in Table 4-10, compute the mean value, assume it is the same as the median, then form Table 4-11 in keeping with whether the values in Table 4-10 are above, a, the median, or b, below it. The mean of data in Table 4-10 is $\bar{x} = 98/50 = 1.96$. Table 4-11 is easy to construct.

TABLE 4-11

DEFINITIVE CATEGORIES OF A-4 EJECTION INJURY DATA

b, b, a, a, a, b, a, a, a, a, b, a, a, b, b, a, a,
a, b, a, a, a, a, b, a, a, b, b, a, b, a, b, b, a,
a, b, a, b, a, b, a, a, b, a, b, b, b, b, b, b.

In Table 4-11,

$$n_1 = 27,\ n_2 = 23, \text{ and } u = 27.$$

where n_1 = number above median, a

n_2 = number below median, b

u = number of runs

From equations (4.23) and (4.24):

$$E(u) = \frac{(2)\ (27)\ (23)}{27 + 23} + 1 = 25.84$$

$$Var(u) = \frac{(2)\ (27)\ (23)\ [2\ (27)\ (23) - 27 - 23]}{(27 + 23)^2\ (27 + 23 - 1)}$$

$$Var(u) = 12.085$$

$$\sqrt{Var(u)} = 3.4764$$

Proceeding as before, compute z_c

$$z_c = \frac{u - E(u)}{\sqrt{Var(u)}}$$

$$= \frac{27 - 25.84}{3.48}$$

$$z_c = 0.33369$$

Since $- z_{0.025} < z_c < z_{0.025}$, that is, since $- 1.96 < 0.33369 < + 1.96$, the hypothesis of randomness cannot be rejected at the level of significance $\alpha = 0.05$, and we conclude that the sample of a's and b's in Table 4-11 have been drawn at random from a single parent population.

4.5.3 The Trend Test

Consider a given data set x_i, $i = 1, 2, \ldots, N$. For example the set of injury data shown in Table 4-10. In this data set, count the number of times that $x_i > x_j$ for $i < j$. This total number of inequalities or reverse arrangements contained in the set $\{x_i\}$ is equal to some number y. If the N data points are independent observations of the same variable x_i, then y is random variable. The variable y has a mean and variance given by:

$$\mu_y = \frac{(N)\ (N - 1)}{4} \tag{4.27}$$

and

$$\sigma_y^2 = \frac{2N^3 + 3N^2 - 5N}{72} \tag{4.28}$$

If N is sufficiently large, then y is approximately normally distributed with mean and variance defined by equations (4.27) and (4.28). Under these conditions, a two-tail level of significance test can be performed on the number of observed trends, here defined as reverse arrangements, detected in a given data set. Given a significance level, α, two points N_1 and N_2 can be determined such that

$$\left.\begin{aligned} \frac{\alpha}{2} &= \int_{-\infty}^{N_1} f(y)\,dy \\ \frac{\alpha}{2} &= \int_{N_2}^{\infty} f(y)\,dy \end{aligned}\right\} \quad f(y) = \frac{1}{\sigma_y\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma}\right)^2} \tag{4.29}$$

Translate the normal density function defined by equation (4.29) to one defined by equation (4.26) by the linear transformation:

$$z = \frac{y - \mu_y}{\sigma_y} \tag{4.30}$$

Clearly,

$$y = -z_{\alpha/2}\,\sigma_y + \mu_y \tag{4.31}$$

$$y = z_{\alpha/2}\,\sigma_y + \mu_y \tag{4.32}$$

In the event $\alpha = 0.05$, the above reduces to (see tables of the standard normal density function, i.e., $\mu_x = 0$, $\sigma_x = 1.0$):

$$N_1 = -1.96\ \sigma_y + \mu_y \tag{4.33}$$

$$N_2 = 1.96\ \sigma_y + \mu_y \tag{4.34}$$

Consider now the data set shown in Table 4-10. Here, N = 50. Hence, from equations (4.27), (4.28), (4.33) and (4.34)

$$\mu_y = \frac{(50)(50 - 1)}{4} = 612.5$$

$$\sigma_y^2 = \frac{(2)(50)^3 + (3)(50)^2 - (5)(50)}{72}$$

$$\sigma_y^2 = 3572.917; \ \sigma_y = 59.774$$

$$N_1 = (-1.96)(59.774) + 612.5 = 495.3$$

$$N_2 = (1.96)(59.774) + 612.5 = 729.7$$

Now postulate the following null hypothesis:

H_o: The A-4 ejection injury profile data (Table 4-10) contain no underlying trend.

To test the null hypothesis, compute the number of trends (reverse arrangements) in the injury profile data. Then, if

$$N_1 \leq \text{(number of observed trends)} \leq N_2$$

accept the null hypothesis at the α level of significance. Otherwise reject it. The number of trends was computed and are displayed in Table 4-12 below.

TABLE 4-12

TREND ANALYSIS OF A-4 EJECTION INJURY DATA
(Sample Extracted from All A-4 Ejections
Over the Time Period 1969-1975)

0, 0, 47, 41, 21, 0, 33, 38, 20, 32, 0, 31, 34, 0, 0, 32, 29, 17, 0, 16, 28, 16, 16, 0, 15, 15, 0, 0, 19, 0, 12, 0, 0, 10, 10, 0, 9, 0, 8, 0, 8, 7, 0, 6, 0, 0, 0, 0, 0, 0.

The total number of trends in Table 4-12 is n = 570. Thus,

$$N_1 < n < N_2 ,$$

or,

$$495.3 < 570 < 729.7 .$$

Therefore, at the $\alpha = 0.05$ level of significance, the null hypothesis of no underlying trend in this sample of A-4 ejection injury data is accepted.

4.5.4 Fisher's Exact Test as Applied to Contingency Tables

The Chi-square goodness-of-fit test gives a good measure of dependency among the various entries in a contingency table if m and n, the row and column dimensions, respectively are large and if the number of elements in each cell (frequency of occurrence) is large. If the number of elements in each cell is small, the Yates' correction factor, defined by equation (4.19) should be applied to enhance a realistic measure of dependency. As an example, consider Table 3-5 repeated here in complete form as Table 4-13 for convenience.

The entries in parenthesis in Table 4-13 are the expected values as computed from equation (4.20). The Chi-square statistic, computed by equation (4.18), is

$$\chi^2_{/d.f.=1} = \frac{(2 - 5.6)^2}{5.6} + \frac{(5 - 1.4)^2}{1.4} + \frac{(38 - 34.4)^2}{34.4}$$

$$+ \frac{(5 - 8.6)^2}{8.6}$$

$$\chi^2_{/d.f.=1} = 13.46$$

TABLE 4-13

A 2 x 2 NECK INJURY CONTINGENCY TABLE

Seat Status → / ↓ Injury Status	NO UPDATE $f_{i,1}$	UPDATE $f_{i,2}$	TOTAL n_i
Neck Injury, $f_{1,j}$	2(5.6)	5(1.4)	7
No Neck Injury $f_{2,j}$	38(34.4)	5(8.6)	43
TOTAL n_j	40	10	50

The value of $\chi^2_{/d.f.=1} = 13.46$. Thus, the hypothesis of independence between neck injuries and updated seats is rejected at the $\alpha = 0.05$ level of significance where $\chi^2_T = 3.841$ for one degree of freedom.

This value of $\chi^2_c = 13.46$ could be seriously in error. Therefore Yates' correction factor, defined by equation (4.19) is applied. This yields a corrected value for Chi-square. Thus,

$$\chi^2_{/d.f.=1} = \frac{(|2 - 5.6| - 1/2)^2}{5.6} + \frac{(|5 - 1.4| - 1/2)^2}{1.4}$$

$$+ \frac{(|38 - 34.4| - 1/2)^2}{34.4} + \frac{(|5 - 8.6| - 1/2)^2}{8.6}$$

$$\chi^2_{/d.f.=1} = 9.9772$$

This corresponds to a level of significance, found by linear interpolation, of $\alpha = 0.00216$. The null hypothesis of independence between neck injuries and updated seats still would be rejected at the given level-of-significance; however, the numerical value for the Chi-square statistic is more realistic.

For a 2 x 2 contingency table, Fisher [7, pg. 230] derived a method for obtaining an exact probability of occurrence among entries in the contingency table. This method also is known as the exact distribution for 2 x 2 contingency tables [10, pg. 96].

In essence, Fisher's Exact Test takes advantage of the Hypergeometric density function defined by the equation:

$$f(x;\, n,\, \zeta,\, \beta) = \frac{\binom{\zeta}{x}\binom{\beta}{n - x}}{\binom{\zeta + \beta}{n}} \tag{4.35}$$

This discrete density function is associated with the physical problem of sampling without replacement.

To apply the above to a 2 x 2 contingency table, consider the table shown in Figure 3-1. Compare contingency table entries with the symbols in equation (4.35) and hence establish the following correspondences:

$$f[a_1;\, (a_1 + b_1),\, (a_1 + a_2),\, (b_1 + b_2)] = \frac{\binom{a_1 + a_2}{a_1}\binom{b_1 + b_2}{b_1}}{\binom{a_1 + a_2 + b_1 + b_2}{a_1 + b_1}} \tag{4.36}$$

Reduce the binomial coefficients to factorials, write the left member of equation (4.36) as $f(a_1)$, then equation (4.36) becomes:

$$f_1\,(a_1) = \frac{(a_1 + a_2)!\;(a_1 + b_1)!\;(a_2 + b_2)!\;(b_1 + b_2)!}{a_1!\;a_2!\;b_1!\;b_2!\;(a_1 + a_2 + b_1 + b_2)!} \tag{4.37}$$

The right member of equation (4.37) is invariant under a permutation of entries in the contingency table. For example:

$$f[b_2;\ (a_2 + b_2),\ (b_1 + b_2),\ (a_1 + a_2)] = \frac{\binom{b_1 + b_2}{b_2}\binom{a_1 + a_2}{a_2}}{\binom{a_1 + a_2 + b_1 + b_2}{a_2 + b_2}} \tag{4.38}$$

The right member of equation (4.38) reduces to the right member of (4.37). Thus, without loss of generality, assume that $a_1 = \min(a_1, a_2, b_1, b_2)$. Equation (4.37) gives the exact probability of observing the two fractions $p_1 = a_1/(a_1 + a_2)$ and $p_2 = b_1/(b_1 + b_2)$ when there is no class difference. That is, under these conditions, the true dichotomy for each class is in the proportion $(a_1 + b_1) : (a_2 + b_2)$. To obtain the final probability used in determining whether a significant difference exists between p_1 and p_2, the probabilities of more divergent fractions than those observed must be added to equation (4.37). The next more divergent situation, assuming $p_1 < p_2$, is obtained by decreasing a_1 and b_2 and increasing a_2 and b_1 each by unity. Thus, analogous to equation (4.37):

$$f_2\ (a_1 - 1) = \frac{(a_1+a_2)!\ (a_1+b_1)!\ (a_2+b_2)!\ (b_1+b_2)!}{(a_1-1)!\ (a_2+1)!\ (b_1+1)!\ (b_2-1)!\ (a_1+a_2+b_1+b_1)!} \tag{4.39}$$

A general expression can now be written:

$$f_i\ (a_1 - i + 1) = \frac{(a_1+a_2)!\ (a_1+b_1)!\ (a_2+b_2)!\ (b_1+b_2)!}{(a_1-i+1)!\ (a_2+i-1)!\ (b_1+i-1)!\ (b_2-i+1)!\ (a_1+a_2+b_1+b_2)!}, \tag{4.40}$$

where $i = 1, 2, \ldots, (a_1 + 1)$.

Finally, the probability of observing two fractions as much or more divergent than p_1 and p_2 when there is no difference in the sampled population is given by [9, pg. 96]:

$$P = \sum_{i=1}^{a_1+1} f_i\ (a_1 - i + 1) \tag{4.41}$$

To apply Fisher's Exact Probability Test to Table 4-13, let $a_1 = 2$, $a_2 = 5$; $b_1 = 38$, and $b_2 = 5$. Then from equation (4.40), when $i = 1$,

$$f_1\ (a_1) = \frac{7!\ 40!\ 10!\ 43!}{2!\ 5!\ 38!\ 5!\ 50!} = 0.0019679$$

Likewise, from equation (4.40), with $i = 2$,

$$f_2\ (a_1 - 1) = \frac{7!\ 40!\ 10!\ 43!}{1!\ 6!\ 39!\ 4!\ 50!} = 0.0000841$$

and

$$f_3\ (a_1 - 2) = f_3\ (0) = \frac{7!\ 40!\ 10!\ 43!}{0!\ 7!\ 40!\ 3!\ 50!} = 0.00000120$$

From equation (4.41), $P = 0.002053$, corresponding to $\chi^2 = 10.0534$, or for a two-tail test, $P = 0.004106$. Thus, the probability of the observed contingency Table 4-13 is 0.0019679, and the probabilities of the other two more extreme cases are 0.0000841 and 0.00000120, respectively.

From the above results, it is seen that Fisher's exact probability one-tail test ($P = 0.002053$) closely approximates the Chi-square goodness-of-fit test to which Yates' correction factor has been applied ($P = 0.00216$); however, the null hypothesis of independence between neck injuries and updated seats is still rejected. To summarize, the preceding null hypothesis is rejected at the $\alpha = 0.05$ level of significance by the following tests: (1) the uncorrected Chi-square test applied to a contingency table, (2) the corrected (Yates' correction) Chi-square test applied to a contingency table, and (3) Fisher's Exact Test.

Section 5.0

SUMMARY

5.0 SUMMARY

This report addresses activities under Phase I of a four phase project to analyze AAES data. Specifically, the charter was to establish a statistical analysis methodology which could be applied to AAES ejection data. It was recognized early that specific analyses may need to be modified, depending on available data--modified in the sense that some analytic techniques might need to be added, others eliminated. In addition, an outline of Phases II, III and IV were to be included in this Phase I report.

This final technical report covering Phase I activities contains an approach to the construction of a statistical analysis methodology for application to AAES ejection data. The derived statistical techniques are applied to actual AAES data. Although primary emphasis was placed on injury data, it should be noted that the displayed techniques are data independent. That is, they can be applied to any data, in general, and to AAES data, in particular, that are amenable to statistical analysis.

The report is heavily illustrated with numerical examples, together with interpretations. In several instances, confidence measures are attached to the particular results obtained. Again, it should be noted that actual AAES ejection data, extracted from both the MORs and combat information, were used throughout this report. Application of the techniques to actual AAES ejection data will continue in Phase II of the investigation.

To reiterate briefly, the direction under which this work is being pursued, is categorized as follows:

- Phase I. Develop and illustrate a statistical analysis methodology applicable to AAES ejection data.
- Phase II. Apply the statistical analysis methodology developed in Phase I to a particular AAES ejection seat.
- Phase III. Expand the specific application illustrated in Phase II to a wide class of ejection seats.

- **Phase IV.** Based on findings and recommendations in Phases II and III, outline a system for monitoring safety aspects of AAES.

Proposed procedures for Phases II, III and IV of this project are outlined in Appendix B, Sections 9, 10 and 11, respectively.

Section 6.0

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6.0 BIBLIOGRAPHY

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APPENDIX A

Section 7.0

CONTINUOUS PROBABILITY DENSITY FUNCTIONS

APPENDIX A: ADDITIONAL STATISTICAL ANALYSES

7.0 Continuous Probability Density Functions

Selected discrete probability density functions were mentioned earlier. It was shown that the binomial density function is useful in the analysis of a dichotomous situation. The Poisson is useful in certain applications of the binomial density function. It is also very useful in many analyses of system reliability problems. The multinomial probability density function is useful if a situation can assume polychotomous states. Examples illustrating use of each of these density functions were displayed.

Here, a brief presentation will be made of three useful continuous probability, density functions: (1) Normal, (2) Gamma, and (3) Beta. The normal density function is displayed because it can be viewed as the foundation of modern statistical theory. The gamma contains the exponential density function as a special case, hence it is useful not only in classical statistical analysis, but in the field of reliability as well. An example of goodness-of-fit to the gamma density function will be given. Also, an example of numerically integrating the gamma density function is shown. The beta density function has the uniform density function as a special case. Further, it evolves quite naturally from the gamma density function. The beta density function is useful in Bayesian statistics.

7.1 The Normal Probability Density Function

The normal density function is useful in data analysis because it is used to tell whether a particular phenomenon, reflected by the data, occurred randomly, that is, by chance. If the event occurred randomly, clearly there was a cause underlying the occurrence, and that cause may be very important. If the event did not occur randomly, then by elimination, there is an underlying deterministic reason for occurrence of the phenomenon. If deterministic causes are suspected, more intensive analyses need to be performed to discover what these are. After discovering the causes, recommendations need to be made to management. As an example, the normal density function

can be used to determine whether a given class of injuries occurred randomly in a given chronological sequence of ejections.

The normal probability density function, sometimes called the Gaussian probability density function in honor of Karl Gauss (1777-1855), is bell-shaped in appearance. A graph of this function is shown in Figure 7-1.

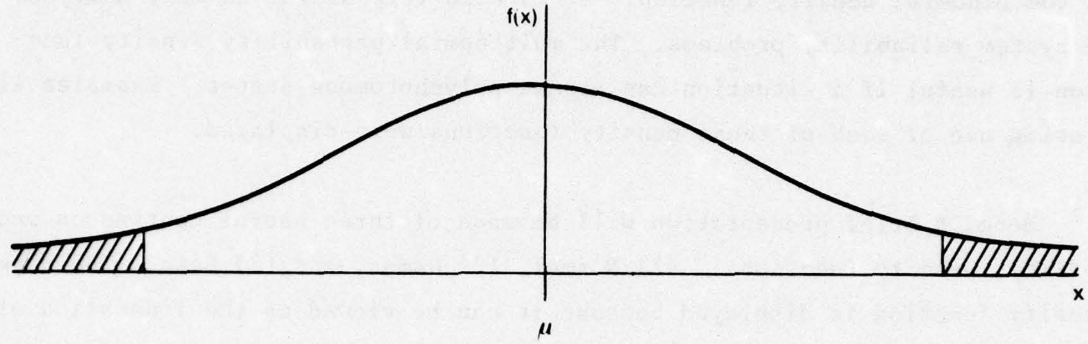

FIGURE 7-1. GRAPH OF THE NORMAL PROBABILITY DENSITY FUNCTION

The equation of a general normal probability density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; -\infty < x < +\infty \tag{7.1}$$

where

σ^2 = variance of the parent density function

μ = mean of the parent density function

x = a continuous independent variable

$f(x)$ = a continuous dependent variable.

This function has been demonstrated to be a probability density function by many investigators. Part of their investigation involves showing that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.0.$$

The cumulative distribution function is defined as follows:

$$F(y) = \int_{-\infty}^{y} f(x)\, dx, \tag{7.2}$$

and the probability that $a < x < b$ is found as follows:

$$P(a < x < b) = \int_{x=a}^{b} f(x)\, dx. \tag{7.3}$$

From equation (7.3), it is clear that $P(x = a) \approx 0$, as is true for other continuous probability density functions. The normal probability will be used to test various hypotheses of random phenomena in the AAES ejection data.

7.2 The Gamma Probability Density Function

The gamma probability density function can be inferred from the gamma function which, in general form, is defined by the *expression:*

$$\Gamma(\alpha) = \frac{1}{\beta^{\alpha}} \int_{x=0}^{\infty} x^{\alpha-1}\, e^{-x/\beta}\, dx. \tag{7.4}$$

From equation (7.4), the gamma probability density function immediately emerges:

$$f(x) = \frac{1}{\Gamma(\alpha)\, \beta^{\alpha}}\, x^{\alpha-1}\, e^{-x/\beta}\ ; \quad \begin{cases} 0 < x < +\infty \\ \alpha > 0,\ \beta > 0 \end{cases} \tag{7.5}$$

Upon inspection of equations (7.4) and (7.5), it is seen that

$$\int_{0}^{\infty} f(x)\, dx = 1.0,$$

where f(x) is defined by equation (7.5). A graph of the gamma probability density function is shown in Figure 7-2.

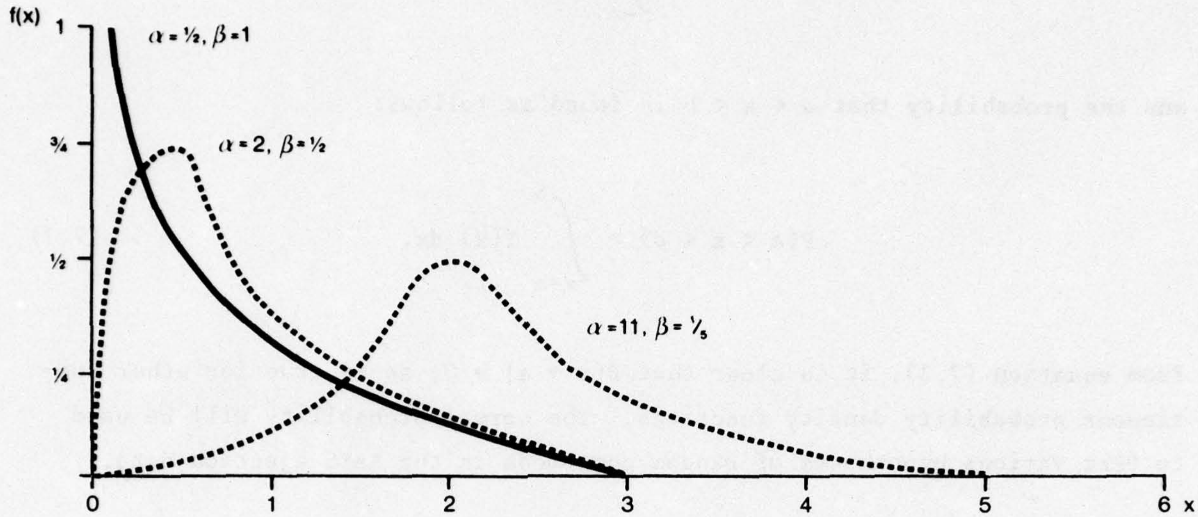

FIGURE 7-2. GRAPH OF THE GAMMA PROBABILITY DENSITY FUNCTION FOR VARIOUS VALUES OF α AND β

In equation (7.5), it should be noted that the exponential probability density function is derived whenever $\alpha = 1$. Thus,

$$f_1(x) = \frac{1}{\beta} e^{-x/\beta} \quad ; \begin{cases} 0 \leq x < +\infty \\ \beta > 0 \end{cases} \tag{7.6}$$

The probability density function defined by equation (7.6) is used quite often with success in reliability and maintainability investigations. Here, "success" is defined as the discovery of a parent probability density function. The importance of such a discovery resides in the predictive capability which it gives an analyst. Additional exponential probability density functions such as the Weibull density function may be employed in the AAES study if they are deemed appropriate. The Weibull density function is defined by the equation:

$$f_2(x;\ \alpha,\ \beta,\ \gamma) = \frac{x - \alpha}{\beta}\ e^{-\left(\frac{x-\alpha}{\beta}\right)^{\gamma}} \tag{7.7}$$

In equation (7.5), note that f(x) has a singularity at the origin whenever $0 < \alpha < 1$. Further, if $\alpha = (n-1)/2$ and $\beta = 2$, the Chi-square probability density function, to be discussed presently in Section 8, emerges. The normal probability density function can be derived by letting $x = z^2$, $\alpha = 3/2$, and $\beta = 1$. Then from equation (7.4),

$$f_3(z) = \frac{1}{\sqrt{\pi}}\ e^{-z^2} \tag{7.8}$$

From equation (7.5) it can be shown that the cumulative distribution function is:

$$F(x) = \int_{y=0}^{x} f(y)\ dy.$$

$$F(x) = \frac{1}{\Gamma(\alpha)\ \beta^{\alpha}} \int_{y=0}^{x} y^{\alpha-1}\ e^{-y/\beta}\ dy \tag{7.9}$$

Similarly,

$$P(a < x < b) = \int_{x=a}^{b} f(x)\ dx = 1 - \alpha \tag{7.10}$$

Finally,

$$\int_{x=0}^{a} f(x)\ dx = \alpha/2 = \int_{x=b}^{\infty} f(x)\ dx, \tag{7.11}$$

where f(x) is the gamma probability density function defined by equation (7.5). Equation (7.11) is used whenever a hypothesis is being tested using a two-tailed test. It is noted in equation (7.11), that α represents level-of-significance, not one of the parameters in the gamma probability density function.

7.2.1 Example of Goodness-of-Fit of Data to the Gamma Density Function

Goodness-of-Fit can be measured with the Chi-square statistic, which will be discussed in more detail in Section 8. As an example of application of goodness-of-fit, consider the data in Table 4-3. Note that there were a total of 107 ejection related injuries. Normalize these injuries by dividing each by 107, form a frequency histogram, and graph the results as shown in Figure 7-3. The normalized data are ranked according to increasing severity of injury. Now assume validity of the following null hypothesis:

H_o: The sample of ejection injuries shown in Figure 7-3 is a random sample extracted from a continuous parent density function that is gamma distributed.

Test of this hypothesis can be performed with the Chi-square statistic defined by the equation:

$$\chi^2/_{d.f.=k-1} = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i} , \qquad (7.12)$$

where

o_i = observed frequency,
e_i = expected (theoretical) frequency,
k = number of frequency cells into which the data have been grouped,
d.f. = degrees of freedom,
χ^2 = Chi-square statistic.

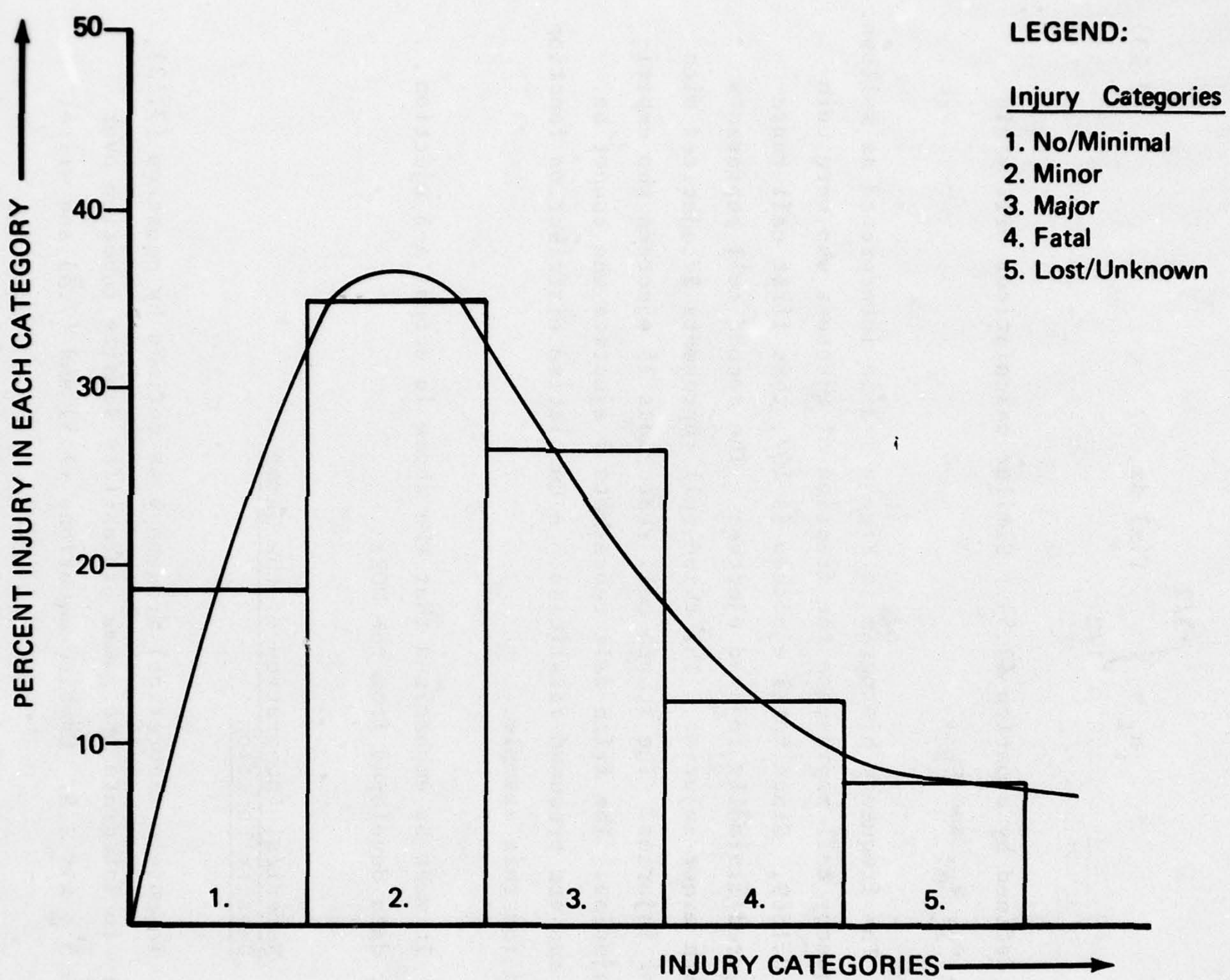

FIGURE 7-3. EXAMPLE OF GOODNESS-OF-FIT

In equation (7.12), the observed frequencies are noted as follows: $o_1 = 0.1869$, $o_2 = 0.3552$, $o_3 = 0.2523$, $o_4 = 0.1402$, and $o_5 = 0.0654$. Expected frequency is determined as follows:

$$e_1 = \int_{1/2}^{3/2} f(x)\, dx, \qquad (7.13)$$

where f(x) is defined by equation (7.5). Similar calculations will yield values for e_2, e_3, e_4, and e_5.

The frequency histogram in Figure 7-3 is interpreted as follows: The first frequency cell represents the fraction of ejectees who were uninjured, namely 0.1869. Since total ejectees is 107, this first cell represents 20 uninjured/minimally injured ejectees. The second cell represents 38 ejectees with minor injuries. The third cell represents 27 ejectees with non-fatal major injuries. The fourth cell represents 15 ejectees who experienced fatal injuries. The fifth cell represents 7 ejectees who cannot be accounted for and are presumed fatalities. A cumulative distribution function could be drawn for this example.

It must be emphasized that the above is actual A-6 ejection related injury data developed from the MORs.

7.2.2 Numerical Integration of the Gamma Density Function

To obtain theoretical frequency as defined by equation (7.13), it is necessary to integrate the gamma probability density function over the interval $0.5 \leq x \leq 1.5$. Combine equations (7.3) and (7.5) and write:

$$P(a < x < b) = \frac{1}{\Gamma(\alpha)\, \beta^{\alpha}} \int_{x=a}^{b} x^{\alpha-1}\, e^{-x/\beta}\, dx. \qquad (7.14)$$

There are two cases to study: (1) α is an integer, and (2) α is not an integer.

Case 1. <u>α is an integer</u>

For this case, equation (7.14) can be written

$$F(b) - F(a) = \frac{1}{\Gamma(\alpha)\,\beta^{\alpha}}\left[\beta a^{\alpha-1}\,e^{-a/\beta} - \beta\, b^{\alpha-1}\,e^{-b/\beta} + \beta\,(\alpha-1)\int_{x=a}^{b} x^{\alpha-2}\,e^{-x/\beta}\,dx\right] \tag{7.15}$$

From equation (7.15) it is easy to observe that eventually $x^{\alpha-k} = 1$ whenever $k = \alpha$, so that a terminating expression exists for $P\ (a < x < b)$. Generally, the problem of primary interest is that of finding b such that a prescribed levels-of-significance is satisfied.

Case 2. <u>α is a not an integer</u>

Perhaps the easiest way to compute theoretical frequency, whenever α is not an integer is to expand $e^{-x/\beta}$ in a power series, multiply by $x^{\alpha-1}$, integrate term by term, insert limits and simplify to get:

$$P(a < x < b) = \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\left[\frac{(-1)^{n}\,(b^{\alpha+n} - a^{\alpha+n})}{(\alpha+n)\,\beta^{\alpha+n}\,n!}\right] \tag{7.16}$$

The right side of equation (7.16) can be shown to converge by the ratio test.

To consider a specific numerical example, let $a = 0$, $b = 46$, $\alpha = 5.236$, and $\beta = 2$. Then for $n = 70$, the numerical value for the 71^{st} term is:

$$u_{70} = \frac{23^{75.236}}{(4.236)\,(3.236)\,(2.236)\,(1.236)\,\Gamma(1.236)\,(75.236)\,(70!)}$$

The $\Gamma(1.236)$ can be looked-up in tables such as [6], or computed from equation (7.4). Likewise, for large $n > 20$; $n!$ can be computed from Stirling's approximation

$$n! = \sqrt{2\pi\, n}\, (n/e)^n \tag{7.17}$$

Thus for $n = 70$, $\Gamma(1.236) = 0.90964$, $70! = 1.196432 \times 10^{100}$, hence

$$u_{70} = 0.0910577.$$

Likewise,

$$u_{69} = -\, 0.28053164.$$

In a similar fashion, $u_0, u_1, \ldots, u_{68}$, etc. could be computed and an approximation written:

$$P(0 < x < 46) = \frac{1}{\Gamma(5.236)} \sum_{n=0}^{70} u_n\, , \tag{7.18}$$

where u_n is the summand in equation (7.16), thus,

$$u_n = \frac{(-\,1)^n\, (b^{\alpha+n} - a^{\alpha+n})}{(\alpha + n)\, \beta^{\alpha+n}\, n!} \tag{7.19}$$

The above numerical process can be applied to the earlier example to get expected (theoretical) frequencies. Knowing o_i's from observation and e_i's from numerical integration enables one to compute the Chi-square statistic, defined by equation (7.12) to confirm, or not, the null hypothesis at a given level-of-significance. The computation shown above merely illustrates a method for numerically calculating theoretical frequency. No goodness-of-fit calculations are involved.

7.3 The Beta Probability Density Function

The beta probability density function has found important applications in Bayesian statistics in recent years where probabilities are sometimes viewed as random variables, and there exists a need for a flexible probability density function which assumes non-zero values only on the interval $0 < x < 1$. Here, flexibility implies that the beta density function can assume a wide variety of shapes. As appropriate, the beta probability density function will be used with Bayesian statistics, discussed in Section 4, to compute refined estimates of prior probability estimates on the occurrence of various aircraft operational events related to this AAES study.

The uniform distribution

$$f(x) = \begin{cases} 1 \; ; \; 0 < x < 1 \\ 0 \; ; \; \text{elsewhere} \end{cases} \tag{7.20}$$

is a special case of the beta probability density function. The beta parent probability density function is defined as follows:

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, x^{\alpha-1} (1 - x)^{\beta-1} \; ; \; 0 < x < 1 \\ 0 \; ; \; \text{elsewhere} \end{cases} \tag{7.21}$$

In equation (7.21), $\alpha > 0$ and $\beta > 0$. Note that the uniform density function can be obtained from equation (7.21) by letting $\alpha = \beta = 1$.

The incomplete beta probability density function is defined thus:

$$B_y (\alpha, \beta) = \int_0^y f(x)\, dx; \quad 0 \leq x \leqq y < 1, \tag{7.22}$$

where $f(x)$ is the beta probability density function defined by equation (7.21).

Levels-of-significance and confidence intervals for the beta density function can be defined in a way similar to definitions for these concepts already displayed for other probability density functions. Thus,

$$\int_0^a f(x)\,dx = \int_b^1 f(x)\,dx = \alpha/2,$$

where $0 < a < b < 1$, and

$$P(a < x < b) = \int_a^b f(x)\,dx = 1 - \alpha \tag{7.23}$$

It should be emphasized that the α in equation (7.23) represents level-of-significance rather than a parameter in either the gamma or beta probability density functions.

APPENDIX A

Section 8.0

SMALL SAMPLE STATISTICAL ANALYSIS: SAMPLING PROBABILITY DENSITY FUNCTIONS

APPENDIX

8.0 Small Sample Statistical Analysis: Sampling Probability Density Functions

It is true in many statistical analyses that relatively large samples, that is samples having many elements, are studied. Under those conditions, various valid simplifying approximations can be made. Thus, discrete density functions such as binomial and Poisson approach the normal density function; the Stirling approximation to n! can be used with good accuracy; and the mean and variance for a run test can be computed under the assumption of the approximation of the discrete parent density function to the normal density function.

Frequently, however, data such as AAES ejection data, are obtained only in small sample sizes. Accordingly, small sample statistical theory must be brought to bear on the statistical problem under consideration. This section will present very briefly some pertinent concepts from the field of exact sampling and demonstrate how these can be used in the statistical analysis of AAES data. Numerical examples are given to illustrate various tests performed.

8.1 Joint Frequency Function for Sample Mean and Sample Variance

The joint frequency function for a sample of N independent normal random variates $x_1, x_2, \ldots, x_N$ is

$$f(x_1, x_2, \ldots, x_N) = (2\pi\,\sigma^2)^{-N/2}\, e^{-v^2/2\sigma^2} \tag{8.1}$$

where

$$v^2 = \sum_{i=1}^{N} x_i^2 = \sum_{i=1}^{N} (x_i - \bar{x} + \bar{x})^2 \tag{8.2}$$

$$v^2 = Ns^2 + N\bar{x}^2 ,$$

and

$$\sum_{i=1}^{N} (x_i - \bar{x}) = 0$$

Equation (8.1) shows that $f(x_1, x_2, \ldots, x_N)$ is a function of the sample mean $\bar{x}$ and sample variance s^2. To find the distribution of $\bar{x}$ and s^2 separately, change from the set $x_1, x_2, \ldots, x_N$ to the set $\bar{x}$, s, w_1, w_2, ..., w_{N-2}, where the w_i's are chosen in a way such that the following two conditions hold:

$$\sum_i (x_i - \bar{x}) = 0 \tag{8.3}$$

$$\sum_i (x_i - \bar{x})^2 = Ns^2 \tag{8.4}$$

The transformation from one set of variables to another is performed by means of Jacobians. Thus,

$$dx_1\, dx_2 \ldots dx_N = |J|\, d\bar{x}\, ds\, dw_1 \ldots dw_{N-2} ,$$

where

$$J = J\left(\frac{x_1, x_2, \ldots, x_N}{\bar{x}, s, w_1, \ldots, w_{N-2}}\right)$$

or in expanded form the Jacobian J, as a determinant of the transformation from one space to another, can be written as follows:

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial \bar{x}} & \dfrac{\partial x_1}{\partial s} & \cdots & \dfrac{\partial x_1}{\partial w_{N-2}} \\ \dfrac{\partial x_2}{\partial \bar{x}} & \dfrac{\partial x_2}{\partial s} & \cdots & \dfrac{\partial x_2}{\partial w_{N-2}} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \dfrac{\partial x_N}{\partial \bar{x}} & \dfrac{\partial x_N}{\partial s} & \cdots & \dfrac{\partial x_N}{\partial w_{N-2}} \end{vmatrix} \tag{8.5}$$

The frequency function in the new variables is

$$f_1 (\bar{x}, s, w_1, \ldots , w_{N-2}) = |J| \; f(\bar{x}, s) \; , \tag{8.6}$$

where $f(\bar{x}, s)$ is defined by the right side of equation (8.1), and is a function of $\bar{x}$ and s only. By integrating equation (8.6) over all w_i's, $f(\bar{x}, s)$ is obtained.

The relations between the two sets of variables may be taken as follows:

$$\left.\begin{aligned} x_1 &= \bar{x} + s\,N^{1/2}\,w_o\,w_1\,w_2 \ldots w_{N-3}\,w_{N-2} \\ x_2 &= \bar{x} + s\,N^{1/2}\,w_o\,w_1 \ldots w_{N-3}\,(1 - w_{N-2}^2)^{1/2} \\ x_3 &= \bar{x} + s\,N^{1/2}\,w_o\,w_1 \ldots w_{N-4}\,(1 - w_{N-3}^2)^{1/2} \\ &\cdot \\ &\cdot \\ &\cdot \\ x_{N-1} &= \bar{x} + s\,N^{1/2}\,w_o\,(1 - w_1^2)^{1/2} \\ x_{N-2} &= \bar{x} + s\,N^{1/2}\,(1 - w_o^2)^{1/2} \; . \end{aligned}\right\} \tag{8.7}$$

It is easy to see from equations (8.7) that $\Sigma_i (x_i - \bar{x})^2 = Ns^2$. To show that $\Sigma_i (x_i - \bar{x}) = 0$, the variable w_o, introduced earlier, must be expressible in terms of $w_1, w_2, \ldots, w_{N-2}$. Clearly, this is the case, hence both conditions defined by equations (8.3) and (8.4) are satisfied.

From equation (8.7) certain values can be inserted into the Jacobian J defined by equation (8.5). Thus,

$$J = \begin{vmatrix} 1 & N^{1/2} \Pi_{1,2} & sN^{1/2} w_o \Pi_{1,3} & \ldots & sN^{1/2} w_o \Pi_{1,N} \\ 1 & N^{1/2} \Pi_{2,2} & sN^{1/2} w_o \Pi_{2,3} & \ldots & sN^{1/2} w_o \Pi_{2,N} \\ 1 & N^{1/2} \Pi_{3,2} & sN^{1/2} w_o \Pi_{3,3} & \ldots & sN^{1/2} w_o \Pi_{3,N} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & N^{1/2} \Pi_{N,2} & sN^{1/2} w_o \Pi_{N,3} & \ldots & sN^{1/2} w_o \Pi_{N,N} \end{vmatrix} \tag{8.8}$$

In equation (8.8), $\Pi_{i,j} = \Pi_{i,j}$ (w) only, that is, they represent products of w_i's only. The Jacobian can be written, upon factoring out all the $N^{1/2}$ and s.

$$J = N^{(N-1)/2}\, s^{N-2}\, |D| , \tag{8.9}$$

where $|D|$ is a determinant containing only w's.

From equations (8.1) and (8.6), the frequency function is written thus:

$$f_1(\bar{x}, s, w_1, w_2, \ldots, w_{N-2}) = C_1\, |D| s^{N-2}\, e^{-N(\bar{x}^2+s^2)/2\sigma^2} \tag{8.10}$$

where C_1 is a constant depending only on N and σ. Integrate both sides of equation (8.10) over all $w_1, w_2, \ldots, w_{N-2}$ to get:

$$f_2(\bar{x}, s) = C_2\, e^{-N\bar{x}^2/2\sigma^2}\, s^{N-2}\, e^{-Ns^2/2\sigma^2} \; ; \begin{cases} -\infty < \bar{x} < +\infty \\ 0 \leq s^2 < +\infty \end{cases} \qquad (8.11)$$

The actual integration of equation (8.10) needs only to be indicated. The integration of $|D|$ is a constant and is absorbed into the constant C_2. A numerical value for C_2 is:

$$C_2 = \frac{2\,(N/2\sigma^2)^{N/2}}{\sqrt{\pi}\;\Gamma\left(\frac{N-1}{2}\right)} .$$

8.2 The Probability Density Function for the Sample Mean

Clearly the joint density function defined by equation (8.11) is comprised of two independent density functions: one for $\bar{x}$, the other for s. Each of these is written as follows:

$$f_3(\bar{x}) = C_3\, e^{-N\bar{x}^2/2\sigma^2} \qquad (8.12)$$

and

$$f_4(s) = C_4\, s^{N-2}\, e^{-Ns^2/2\sigma^2} \qquad (8.13)$$

Since

$$\int_{-\infty}^{\infty} f_3\,(x)\, dx = 1.0 ,$$

it follows from performing the integration that

$$C_3 = (N/2\pi\,\sigma^2)^{1/2} \qquad (8.14)$$

Hence,

$$f_3(\bar{x}) = (N/2\pi\,\sigma^2)^{1/2}\, e^{-N\bar{x}^2/2\sigma^2} \qquad (8.15)$$

8.3 The Probability Density Function for Sample Variance

Clearly,

$$f_4(s)\, ds = f_5\,(s^2)\, d(s)^2, \text{ or}$$
$$f_4(s) = f_5\,(s^2)\,(2s), \qquad (8.16)$$

hence write equation (8.13) in terms of s^2 instead of s as follows:

$$f_5\,(s^2) = \frac{C_5\,(s^2)^{\frac{N-2}{2}}}{s}\, e^{-Ns^2/2\sigma^2}$$

$$f_5\,(s^2) = C_5\,(s^2)^{\frac{N-3}{2}}\, e^{-Ns^2/2\sigma^2} \qquad (8.17)$$

Again, note

$$\int_0^\infty f_5\,(s^2)\, d(s^2) = 1.0 \ .$$

From this fact, it is easy to show that

$$C_5 = \frac{\left(\frac{N}{2\sigma^2}\right)^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \ ,$$

and finally, equation (8.17) becomes:

$$f_5 (s^2) = \frac{\left(\frac{N}{2\sigma^2}\right)^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} (s^2)^{(N-3)/2} e^{-Ns^2/2\sigma^2} \tag{8.19}$$

Equation (8.19) defines the probability density function of the variance of a random sample of size N.

8.4 The Mean and Variance of the Sample Mean

From equation (8.12), the constant C_3 is computed as follows:

$$C_3 \int_{-\infty}^{\infty} e^{-N\bar{x}^2/2\sigma^2} d\bar{x} = 1.0 \tag{8.20}$$

Thus,

$$C_3 = \left(\frac{N}{2\pi \sigma^2}\right)^{1/2} \tag{8.21}$$

and equation (8.12) becomes

$$f_3 (\bar{x}) = \left(\frac{N}{2\pi \sigma^2}\right)^{1/2} e^{-N\bar{x}^2/2\sigma^2} \tag{8.22}$$

Now the mean of the sample mean is

$$E(\bar{x}) = \left(\frac{N}{2\pi \sigma^2}\right)^{1/2} \int_{-\infty}^{\infty} \bar{x}\, e^{-N\bar{x}^2/2\sigma^2} d\bar{x}$$

Clearly the function

$$g(\bar{x}) = \bar{x}\, e^{-N\bar{x}^2/2\sigma^2}$$

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is an odd function of $\bar{x}$, that is, $g(-\bar{x}) = -g(\bar{x})$. Since the integral of an odd function over symmetric limits is zero, it follows that

$$E(\bar{x}) = 0 \tag{8.23}$$

To compute the variance of the sample mean, write

$$\mathrm{Var}(\bar{x}) = \nu_2 - \nu_1^2 , \tag{8.24}$$

where ν_2 is the second moment. From equation (8.23), $\nu_1 = 0$, hence $\mathrm{Var}(\bar{x}) = \nu_2$.

$$\mathrm{Var}(\bar{x}) = \left(\frac{N}{2\pi\ \sigma^2}\right)^{1/2} \int_{-\infty}^{\infty} \bar{x}^2\ e^{-N\bar{x}^2/2\sigma^2}\ d\bar{x}$$

Let $z = \bar{x}^2$; $\bar{x} = z^{1/2}$; $dz = 2\bar{x}\ d\bar{x}$

Then,

$$\mathrm{Var}(\bar{x}) = 2\left(\frac{N}{2\pi\ \sigma^2}\right)^{1/2} \int_{0}^{\infty} z\ e^{-Nz/2\sigma^2}\ \frac{dz}{2z^{1/2}}$$

$$\mathrm{Var}(\bar{x}) = \left(\frac{N}{2\pi\ \sigma^2}\right)^{1/2} \int_{0}^{\infty} z^{1/2}\ e^{-Nz/2\sigma^2}\ dz$$

$$= \left(\frac{N}{2\pi\ \sigma^2}\right)^{1/2} \frac{\Gamma(3/2)}{\left(\frac{N}{2\sigma^2}\right)^{3/2}} ;\ \Gamma(3/2) = (1/2)\ \Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

This last expression reduces to

$$\mathrm{Var}(\bar{x}) = \sigma^2/N \tag{8.25}$$

8.5 The Mean and Variance of the Sample Variance

The mean of the sample variance can be obtained from the first order moment generating function. Thus,

$$E(s^2) = \int_0^\infty s^2 f_5(s^2)\, ds^2 \tag{8.26}$$

Substitute equation (8.19) into (8.26) to get:

$$E(s^2) = \int_0^\infty \frac{\left(\frac{N}{2\sigma^2}\right)^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} (s^2)^{\frac{N-3}{2}+1} e^{-Ns^2/2\sigma^2}\, ds^2$$

$$= \frac{\left(\frac{N}{2\sigma^2}\right)^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \int_0^\infty (s^2)^{\frac{N-1}{2}} e^{-Ns^2/2\sigma^2}\, ds^2$$

Since

$$\int_0^\infty (s^2)^{\frac{N-1}{2}} e^{-Ns^2/2\sigma^2}\, ds^2 = \frac{\Gamma\left(\frac{N+1}{2}\right)}{\left(\frac{N}{2\sigma^2}\right)^{\frac{N+1}{2}}},$$

it follows that

$$E(s^2) = \frac{\left(\frac{N}{2\sigma^2}\right)^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \cdot \frac{\Gamma\left(\frac{N+1}{2}\right)}{\left(\frac{N}{2\sigma^2}\right)^{\frac{N+1}{2}}},$$

or

$$E(s^2) = \frac{N-1}{N}\sigma^2 . \tag{8.27}$$

In this derivation, recall that

$$\Gamma\left(\frac{N+1}{2}\right) = \frac{N-1}{2}\,\Gamma\left(\frac{N-1}{2}\right)$$

In a similar fashion,

$$\mathrm{Var}(s^2) = \left(2\sigma^4\right)\left(\frac{N-1}{N^2}\right) \tag{8.28}$$

8.6 The Mean and Variance of the Sample Standard Deviation

From equations (8.16) and (8.17),

$$f_4(s) = 2\,C_5\,s^{N-2}\,e^{-Ns^2/2\sigma^2} \tag{8.29}$$

$$E(s) = \int_0^\infty s f_4(s)\,ds$$

$$E(s) = \int_0^\infty 2\,C_5\,s^{N-1}\,e^{-Ns^2/2\sigma^2}\,ds$$

$$= \frac{2\left(\frac{N}{2\sigma^2}\right)^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \cdot \frac{\Gamma\left(\frac{N-2}{2}+1\right)}{2\left(\frac{N}{2\sigma^2}\right)^{\frac{N-2}{2}+1}}$$

$$E(s) = \frac{\left(\frac{2}{N}\right)^{1/2}\Gamma\left(\frac{N}{2}\right)\sigma}{\Gamma\left(\frac{N-1}{2}\right)} \tag{8.30}$$

The variance of s can be found by use of the moment generating function. Thus,

$$\nu_r = \int_0^\infty s^r f_4(s)\, ds \tag{8.31}$$

Substitute equation (8.29) into (8.31) and get:

$$\nu_r = \frac{2\left(\frac{N}{2\sigma^2}\right)^{\frac{N-1}{2}}}{\frac{N-1}{2}} \int_0^\infty s^{N+r-2}\, e^{-Ns^2/2\sigma^2}\, ds$$

Let $z = s^2$. Then $dz = 2s\, ds$. Substitute above to get:

$$\nu_r = \frac{2\left(\frac{N}{2\sigma^2}\right)^{\frac{N-1}{2}}}{2\,\Gamma\left(\frac{N-1}{2}\right)} \int_0^\infty z^{\frac{N+r-2}{2} - 1/2}\, e^{-Nz/2\sigma^2}\, dz$$

$$\nu_r = \frac{\left(\frac{N}{2\sigma^2}\right)^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \cdot \frac{\Gamma\left(\frac{N-r+1}{2}\right)}{\left(\frac{N}{2\sigma^2}\right)^{\frac{N+r-1}{2}}}$$

This reduces to

$$\nu_r = \left(\frac{2}{N}\right)^{r/2} \cdot \frac{\Gamma\left(\frac{N+r-1}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}\, \sigma^r \tag{8.32}$$

$$\mathrm{Var}(s) = \nu_2 - (\nu_1)^2 = \left\{ \frac{\frac{2}{N}\,\Gamma\left(\frac{N+1}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} - \left[\frac{\left(\frac{2}{N}\right)^{1/2}\,\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} \right]^2 \right\} \sigma^2$$

This last equation reduces to

$$\mathrm{Var}(s) = \frac{1}{N}\left[N-1 - \frac{2\,\Gamma^2\,(N/2)}{\Gamma^2\left(\frac{N-1}{2}\right)} \right] \sigma^2 , \tag{8.33}$$

where

$$\Gamma^2\,(t) \equiv [\Gamma(t)]^2 .$$

An approximate value for Var(s) can be written:

$$\mathrm{Var}(s) = \left[\frac{1}{2N} - \frac{1}{8N^2} - \frac{3}{16N^3} - \ldots \right] \sigma^2 , \tag{8.34}$$

so that approximately,

$$\sigma_s^2 = \mathrm{Var}(s) \doteq \sigma^2/2N \tag{8.35}$$

8.7 The Chi-Square Probability Density Function

The Chi-square probability density function can be obtained from equation (8.19) by choosing $\chi^2 = Ns^2/\sigma^2$. Making this substitution yields the following:

$$f_6\,(\chi^2) = \frac{\left(\frac{\chi^2}{2}\right)^{(N-3)/2} e^{-\chi^2/2}}{2\,\Gamma\left(\frac{N-1}{2}\right)} , \tag{8.36}$$

where, to derive equation (8.36), use is made of the equation

$$f_5(s^2)\ d(s^2) = f_6(\chi^2)\ d\chi^2 \tag{8.37}$$

from which it immediately follows that

$$f_5(s^2) = f_6(\chi^2)\ (N/\sigma^2) \tag{8.38}$$

Equation (8.36) is one form of the Chi-square probability density function.

8.8 The t-Probability Density Function

The t-density function, here presented without derivation is

$$f_n(t) = K_n \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} \quad ; \ n = N - 1 \tag{8.39}$$

where

$$1/K_n = n^{1/2}\ B(n/2,\ 1/2). \tag{8.40}$$

B represents the beta function which can be expressed as follows:

$$B(m,\ n) = \frac{\Gamma(m)\ \Gamma(n)}{\Gamma(m + n)}\ , \tag{8.41}$$

in which $\Gamma(m)$ is the gamma function of m. Combining equations (8.40) and (8.41), it is easy to show that

$$K_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\ \Gamma(n/2)}\ , \tag{8.42}$$

or in terms of N, where n = N - 1,

$$K_N = \frac{\Gamma(N/2)}{\sqrt{(N-1)\pi}\ \Gamma\left(\frac{N-1}{2}\right)} \tag{8.43}$$

Substitute (8.42) into (8.39) to get:

$$f_n(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\ \Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} \tag{8.44}$$

The t-statistic is defined by the expression:

$$t = \frac{(\bar{x} - \mu)\ (N - 1)^{1/2}}{s}\ , \tag{8.45}$$

which shows that $f_n(t)$ is a function only of the population mean, μ. This function, $f_n(t)$, is useful in testing the null hypothesis that a given sample is extracted from a universe with a given mean. Note that $f_n(t)$ is independent of the population variance, σ^2.

It can be shown that for n sufficiently large, the t-density function approaches the normal density function. Recall that for large n, Stirling's approximation gives

$$n! = \sqrt{2\pi\ n}\ (n/e)^n \tag{8.46}$$

Also,

$$\Gamma\left(\frac{n+1}{2}\right) = \left(\frac{n-1}{2}\right)!$$

From this information, K_n is written thus:

$$K_n = \left(\frac{n-1}{n-2}\right)^{1/2} \left(\frac{n-1}{n-2}\right)^{\frac{n-2}{2}} \left(\frac{n-1}{n}\right)^{1/2} \left(\frac{1}{(2\pi\ e)^{1/2}}\right)$$

which immediately simplifies to the following:

$$K_n = \left(\frac{1 - \frac{1}{n}}{1 - \frac{2}{n}}\right)^{1/2} \left(\frac{1 - \frac{1}{n}}{1 - \frac{2}{n}}\right)^{\frac{n}{2} - 1} \left(1 - \frac{1}{n}\right)^{1/2} \left(\frac{1}{2\pi\ e}\right)^{1/2}$$

Hence,

$$\lim_{n \to \infty} K_n = \frac{1}{\sqrt{2\pi}} .$$

Write the variable portion of $f_n(t)$ as follows:

$$g_n(t) = \left(1 + \frac{t^2}{n}\right)^{-1/2} \frac{1}{\left(1 + \frac{t^2}{2} \cdot \frac{1}{\frac{n}{2}}\right)^{n/2}}$$

Clearly,

$$\lim_{n \to \infty} g_n(t) = e^{-t^2/2} .$$

Thus,

$$\lim_{n \to \infty} f_n(t) = \frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2} , \tag{8.47}$$

as was to be shown.

The t-distribution has been found useful in such problems as testing the significance of the difference between two means and testing hypotheses regarding regression coefficients. To illustrate the former concept, let $\bar{x}_1$ and $\bar{x}_2$ be the means and s_1 and s_2 the standard deviations of two independent samples of size N_1 and N_2 variates, respectively. It is assumed that

the samples are extracted from a universe which is normally distributed with mean μ and variance σ^2. The variance of the difference between the two sample means can be shown to be $\sigma^2\ (N_1 + N_2)/N_1\ N_2$ [7, pg. 79]. Hence the t-variable can be expressed as follows:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sigma} \left(\frac{N_1\ N_2}{N_1 + N_2} \right)^{1/2} . \tag{8.48}$$

In practice, an estimator is used in place of σ. The one used is

$$\sigma^2 = \hat{\sigma}^2 = \frac{N_1\ s_1^2 + N_2\ s_2^2}{N_1 + N_2 - 2} \tag{8.49}$$

Substitute equation (8.49) into (8.48) to get:

$$t_o = \frac{\bar{x}_1 - \bar{x}_2}{\left(\frac{s_1^2}{N_2} + \frac{s_2^2}{N_1} \right)^{1/2}} \cdot \left(\frac{N_1 + N_2 - 2}{N_1 + N_2} \right)^{1/2} \quad ; \ n = N_1 + N_2 - 2 \tag{8.50}$$

Several special cases are an immediate consequence of equation (8.50). Thus, if N_1 and N_2 are sufficiently large, equation (8.50) becomes:

$$t_1 = \frac{\bar{x}_1 - \bar{x}_2}{\left(\frac{s_1^2}{N_1} + \frac{s_2^2}{N_1} \right)^{1/2}} \ ; \ n = N_1 + N_2 - 2 \ , \tag{8.51}$$

where n can be viewed as the number of degrees of freedom in the t-statistic.

If $N_2 >> N_1$, then equation (8.50) reduces to the following whenever $\bar{x}_2 \rightarrow \mu$ and $s_1 \rightarrow \sigma$:

$$t_2 = \frac{(\bar{x}_1 - \mu)}{\sigma} \sqrt{N_1} \; ; \; n = N_1 + N_2 - 2 \tag{8.52}$$

If $N_1 = N_2 = N$, then equation (8.50) becomes:

$$t_3 = \frac{\bar{x}_1 - \bar{x}_2}{\left(s_1^2 + s_2^2\right)^{1/2}} \sqrt{N - 1} \; ; \; n = N_1 + N_2 - 2 \tag{8.53}$$

Now consider an example of the use of the t-statistic. The data used in this example are actual ejection related injury data extracted from the MORs.

The following table represents random samples of ejection related injury data upon ejection from: (1) A-4 aircraft with ESCAPAC ejection systems, and (2) A-6 aircraft with Martin-Baker ejection systems. For these two random samples, is there a significant difference in ejection related injury history using the difference between their means as a judgment criterion?

Injury categories in Table 8-1 are the same as those defined in Table 4-3. From the data in Table 8-1, the following numbers are developed:

A-4 Injury Data	A-6 Injury Data
$\sum_{i=1}^{18} x_i = 51$	$\sum_{i=1}^{16} x_i = 48.0$
$\bar{x}_1 = 2.83334$	$\bar{x}_2 = 3.0$
$s_1^2 = 2.264706$	$s_2^2 = 2.133334$
$s_1 = 1.504894$	$s_2 = 1.460593$
$n_1 = 18$	$n_2 = 16$

TABLE 8-1

APPLICATION OF THE t-STATISTIC TO EJECTION RELATED INJURY DATA FROM A-4 AND A-6 AIRCRAFT
(Sample Extracted From MORs Data Over 1969-1975)

Sample 1 A-4 Injury Data	Sample 2 A-6 Injury Data
1	4
3	2
5	3
4	2
2	5
1	2
4	4
1	1
4	3
4	4
1	5
5	2
1	5
3	4
4	1
1	1
3	
4	
$\Sigma = 51.0;\ n_1 = 18$	$\Sigma = 48.0;\ n_2 = 16$

Substitute these numbers into equation (8.50):

$$t_o = \frac{2.83334 - 3.0000}{\left(\dfrac{2.264706}{16} + \dfrac{2.133334}{18}\right)^{1/2}} \cdot \left(\frac{18 + 16 - 2}{18 + 16}\right)^{1/2}$$

$$t_o = -\ 0.31706$$

For a level of significance $\alpha = 0.05$, and n = 32 degrees of freedom,

$$t_{0.025/d.f.=32} = 2.0378$$

Likewise,

$$t_{-0.025/d.f.=32} = -\ 2.0378$$

Since $t_{-\alpha/2} < t_o < t_{+\alpha/2}$, that is, since $-2.0378 < -0.31706 < 2.0378$, the null hypothesis that the two samples studied are extracted from parent universes having the same mean is accepted at the $\alpha = 0.05$ level of significance. Thus, we reject the hypothesis that a significant difference exists between A-4 and A-6 ejection related injury patterns, using the t-test as a judgment criterion.

8.9 The F-Probability Density Function

The F-probability density function is used to test the hypothesis that two random samples of size N_1 and N_2, having variances s_1^2 and s_2^2, are extracted from the same parent density function which has variance σ^2. A detailed study of the derivation of the F-parent density function, as a special case of the Pearson Type IV curve [7, pg. 105], is outside the scope of this paper. An excellent description of arguments showing how the F-density function is derived from Fisher's z-distribution can be found in [7, pp. 180-182]. Let

$$s_1^2 = \frac{1}{N_1 - 1} \sum_{i=1}^{N} (x_{1i} - \bar{x}_1)^2 = \frac{1}{n_1} \sum_{i=1}^{n_1+1} (x_{1i} - \bar{x})^2 \tag{8.54}$$

$$s_2^2 = \frac{1}{N_2 - 1} \sum_{i=1}^{N_2} (x_{2i} - \bar{x}_2) = \frac{1}{n_2} \sum_{i=1}^{n_2+1} (x_{2i} - \bar{x})^2 \tag{8.55}$$

represent variances for sample 1 and sample 2 extracted from a parent density function having population variance σ^2. Since $n_1 = N_1 - 1$ and $n_2 = N_2 - 1$, an unbiased estimator of the population variance from sample 1 is

$$\hat{\sigma}_1^2 = \frac{N_1\ s_1^2}{n_1}\ ;\ n_1 = N_1 - 1 \tag{8.56}$$

and for sample 2

$$\hat{\sigma}_2^2 = \frac{N_2 \, s_2^2}{n_2} \; ; \; n_2 = N_2 - 1 \tag{8.57}$$

To test the hypothesis

H_o: $\hat{\sigma}_1^2 = \hat{\sigma}_2^2 = \sigma^2$, construct the statistic:

$$F = \frac{N_1 \, s_1^2/n_1}{N_2 \, s_2^2/n_2} \tag{8.58}$$

Clearly,

$$\frac{n_1 \, F}{n_2} = \frac{N_1 \, s_1^2/\sigma^2}{N_2 \, s_2^2/\sigma^2} \tag{8.59}$$

in which $N_1 \, s_1^2/\sigma^2$ is χ_1^2 with $(N_1 - 1)$ degrees of freedom and $N_2 \, s_2^2/\sigma^2$ is χ_2^2 distributed with $(N_2 - 1)$ degrees of freedom. The variable $n_1 \, F/n_2$ is distributed according to a Pearson Type IV curve mentioned earlier. The expression is:

$$f(F) = \left(\frac{n_1 \, F}{n_2}\right)^{\frac{n_1-2}{2}} \left(1 + \frac{n_1 \, F}{n_2}\right)^{-\frac{n_1+n_2}{2}} \left(\frac{n_1}{n_2}\right) \Big/ B\left(\frac{n_1}{2}, \frac{n_2}{2}\right), \tag{8.60}$$

in which

$$B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$$

is the beta function defined earlier. Equation (8.60) can be written:

$$f(F) = K\,F^{(n_1-2)/2}\left(1 + \frac{n_1 F}{n_2}\right)^{\frac{n_1+n_2}{2}} \; ; \; 0 \le F < +\infty \tag{8.61}$$

where

$$K = \left(\frac{n_1}{n_2}\right)^{n_1/2} \Bigg/ B\left(\frac{n_1}{2}\,,\,\frac{n_2}{2}\right) \tag{8.62}$$

To illustrate use of the F-statistic, consider two samples of sizes $n_1 + 1$ and $n_2 + 1$ from normal populations with means μ_1, μ_2 and variances

$$\sigma_1^2, \sigma_2^2 ,$$

respectively. Define

$$\theta = \sigma_1^2/\sigma_2^2 ,$$

and let the hypothesis be

H_o: $\theta = 1$, regardless of the actual values of

$$\sigma_1^2, \sigma_2^2, \mu_1 \text{ and } \mu_2 .$$

Let A_{12} and B_{12} be two numbers, depending on n_1 and n_2, chosen so that for any given n_1 and n_2.

$$F = \int_{A_{12}}^{B_{12}} f(F)\, dF = 1 - \alpha \tag{8.63}$$

where α is the level of significance. The geometry is shown in Figure 8-1.

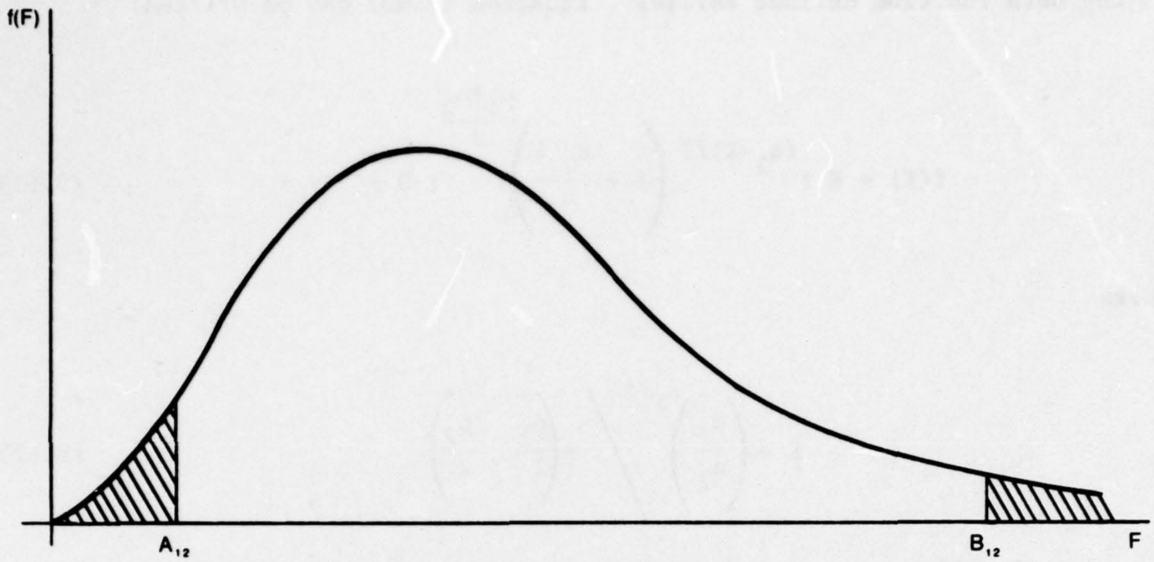

FIGURE 8-1. GRAPH OF THE F-PROBABILITY DENSITY FUNCTION

Another way of writing equation (8.63) is

$$\int_0^{A_{12}} f(F)\, dF = \int_{B_{12}}^{\infty} f(F)\, dF = \alpha/2 \tag{8.64}$$

From equation (8.64), it follows that $A_{12} = 1/B_{21}$, where B_{21} is the number obtained from B_{12} by interchanging n_1 and n_2. Now, define $u = 1/F$, then equation (8.64) can be written:

$$\int_0^{A_{12}} f(F)\, dF = \frac{(n_2/n_1)^{n_2/2}}{B\,\frac{n_1}{2},\,\frac{n_2}{2}} \int_{1/A_{12}}^{\infty} u^{(n_2-2)/2} \left(1 + \frac{n_2\; u}{n_1}\right)^{-\frac{n_1+n_2}{2}} du \tag{8.65}$$

so that $B_{21} = 1/A_{12}$, since the right side of (8.65) is the F-integral with n_1 and n_2 interchanged.

Due to the relationship between A_{12} and B_{12}, it makes no difference which sample is labeled number 1 and which is labeled number 2. With one arrangement, H_o is rejected unless $A_{12} \leq F \leq B_{12}$. With the other arrangement, H_o is rejected unless $A_{21} \leq 1/F \leq B_{21}$. But $B_{21} = 1/A_{12}$ and $A_{21} = 1/B_{12}$. From this, it is clear that conditions of rejection are the same.

To apply the above to a numerical example, consider the injury data, extracted from the MORs, shown in Table 8-1. Thus, $N_1 = 18$, $N_2 = 16$; $n_1 = N_1 - 1 = 17$, $n_2 = N_2 - 1 = 15$;

$$s_1^2 = 2.264706; \; s_2^2 = 2.133334.$$

The null hypothesis is

$$H_o: \; \sigma_1^2/\sigma_2^2 = \theta = 1, \text{ versus the alternate hypothesis}$$

$$H_1: \; \sigma_1^2 > \sigma_2^2 \; .$$

Substituting the numbers developed into equation (8.58) yields:

$$F_o = \frac{N_1 \; s_1^2/n_1}{N_2 \; s_2^2/n_2}$$

Hence,

$$F_o = \frac{(18)(2.264706)/17}{(16)(2.133334)/15}$$

$$F_o = 1.053774$$

Since $F_T(17, 15) = 2.32$, from tabulated values of the F-statistic at $\alpha = 0.05$, the null hypothesis H_o is accepted, and we conclude that the two samples under investigation were extracted from parent density functions having the same variance.

8.10 Confidence Limits

Upon applying the statistical techniques outlined above, it is highly desirable to be able to measure the confidence invested in the computed results. This measure evolves from a study of confidence limits. These will be discussed and applied as appropriate.

8.10.1 Confidence Limits for the Mean

From equation (8.45), the t-statistic is defined as follows

$$t = \frac{(\bar{x} - \mu)\sqrt{n}}{s},$$

where, as before, n = N - 1. To illustrate the discussion that follows, consider Figure 8-2 which is a sketch of the t-density function, showing (1) acceptance region, (2) rejection region (cross-hatched area), and (3) a t-statistic, t_o, computed from some data. From Figure 8-2, it is clear that t_o is in the acceptance region, since $-t_{\alpha/2} < t_o < t_{\alpha/2}$. From the definition of the t-statistic, it is clear that the above inequality can be written:

$$-t_{\alpha/2} < \frac{\bar{x} - \mu}{s/\sqrt{n}} < t_{\alpha/2} \tag{8.66}$$

which easily reduces to

$$\bar{x} - \frac{s}{\sqrt{n}} t_{\alpha/2} < \mu < \bar{x} + \frac{s}{\sqrt{n}} t_{\alpha/2} \tag{8.67}$$

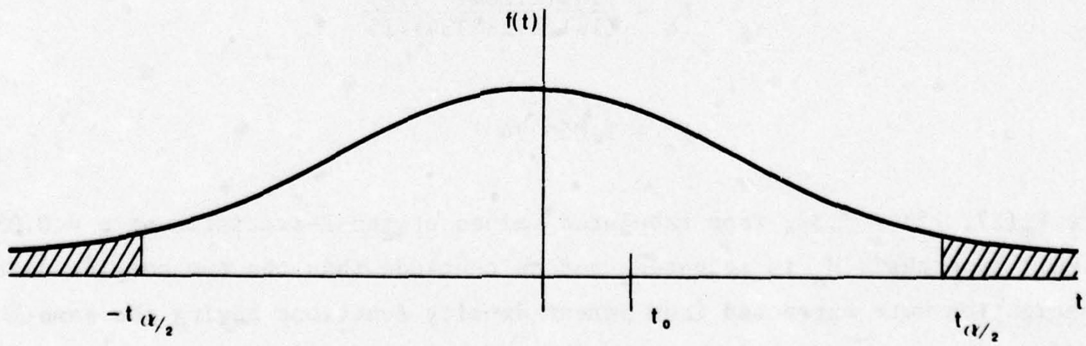

FIGURE 8-2. GRAPH OF THE t-PROBABILITY DENSITY FUNCTION

Inequality (8.67) can be read as follows: The parent population mean, μ, will lie between the limits

$$\bar{x} - \frac{s}{\sqrt{n}} t_{\alpha/2} \text{ and } \bar{x} + \frac{s}{\sqrt{n}} t_{\alpha/2},\ 100 \cdot (1 - \alpha)$$

percent of the time. Compactly written this becomes:

$$P\left(\bar{x} - \frac{s}{\sqrt{n}} t_{\alpha/2} < \mu < \bar{x} + \frac{s}{\sqrt{n}} t_{\alpha/2}\right) = 1 - \alpha\ , \qquad (8.68)$$

where P ( ) means the probability of ( ).

To apply this to a concrete example, consider the ejection injury data shown in Sample 1 of Table 8-1. Two questions need to be addressed: (1) assume the data are extracted from a normal parent density function with mean μ and variance σ^2. What are the bounds on the parent mean, μ, and (2) what confidence can be invested in this assertion? Let $\alpha = 0.05$, and d.f. = 17. Compute the t-statistic thus:

$$t_o = \frac{\bar{x}_1 - \mu}{s_1/\sqrt{n}} = \frac{2.833334 - \mu}{1.504894/\sqrt{17}}$$

From tabulated values of the t-statistic,

$$t_{\alpha/2} = t_{0.025/d.f.=17} = 1.740$$

Substitute into equation (8.67) to get:

$$2.83334 - \left(\frac{1.504894}{\sqrt{17}}\right)(1.740) < \mu < 2.83334 + \left(\frac{1.504894}{\sqrt{17}}\right)(1.740)$$

This simplifies to:

$$2.19825 < \mu < 3.46842 \tag{8.69}$$

The answer to the second question is easy:

$$P(2.19825 < \mu < 3.46842) = 1 - 0.05 = 0.95 \tag{8.70}$$

Thus, it can be said with 95% confidence that the parent density function from which this particular sample was extracted, has a mean μ that will lie between 2.19825 and 3.46842 about 95% of the time.

8.10.2 Confidence Limits for the Difference Between Two Means

To get confidence limits for the difference between two means, first consider equation (8.50) and rearrange it slightly as follows:

$$t_o = \frac{\bar{x}_1 - \bar{x}_2}{\left[\left(\frac{N_1 s_1^2 + N_2 s_2^2}{N_1 + N_2 - 2}\right)\left(\frac{N_1 + N_2}{N_1 N_2}\right)\right]^{1/2}} \tag{8.71}$$

Now make the following definition:

$$\bar{w} = \bar{x}_1 - \bar{x}_2 = (\bar{x}_1 - \mu_1) - (\bar{x}_2 - \mu_1)$$

$$\omega = \mu_1 - \mu_2 \tag{8.72}$$

$$\hat{\sigma}_{\bar{w}} = \left[\left(\frac{N_1 s_1^2 + N_2 s_2^2}{N_1 + N_2 - 2}\right)\left(\frac{N_1 + N_2}{N_1 N_2}\right)\right]^{1/2}$$

The new t-statistic becomes

$$t_o = \frac{(\bar{x}_1 - \mu_1) - (\bar{x}_2 - \mu_2)}{\hat{\sigma}_{\bar{w}}},$$

or

$$t_o = \frac{\bar{w} - \omega}{\hat{\sigma}_{\bar{w}}} \tag{8.73}$$

Equation (8.73) was derived under the following conditions: Let $\bar{x}_1$ and s_1^2 be the observed mean and variance of a random sample of size N_1 drawn from a normal universe with unknown mean μ_1. Also, let $\bar{x}_2$ and s_2^2 be the observed mean and variance of a random sample of size N_2 drawn from a normal universe with unknown mean μ_2. It is assumed that the two universes have the same variance. Equation (8.73) results from the preceding sampling information.

To evaluate confidence limits for the difference between two means, consider again the ejection related injury data in Table 8-1, and substitute those numbers into equation (8.73):

$$t_o = \frac{(2.83334 - 3.00) - (\mu_1 - \mu_2)}{\left[\frac{(18)(2.264706) + (16)(2.13334)}{(18)(16)}\right]^{1/2}} \cdot \left(\frac{32}{34}\right)^{1/2}$$

Choose $\alpha = 0.05$. Then,

$$-t_{\alpha/2} < t_o < t_{\alpha/2} \quad \text{(32 degrees of freedom)}$$

$$-\ 2.0378 < t_o < +\ 2.0378$$

Combine the above to get:

$$-\ 0.90452 < (\mu_1 - \mu_2) < 1.23785 \tag{8.74}$$

From inequality (8.74), it follows immediately that,

$$P(-0.90452 < (\mu_1 - \mu_2) < 1.23785) = 0.95 \tag{8.75}$$

This says that the difference between the means of the parent density functions will lie between - 0.90452 and 1.23785 about 95% of the time.

8.10.3 Confidence Limits for the Variance

To compute confidence limits for the parent variance, first recall equations (8.56) and (8.57) and note that $N_1 s_1^2/\sigma^2$ is χ^2 distributed with $(N_1 - 1)$ degrees of freedom. Thus,

$$\frac{N_1 s_1^2}{\chi_2^2} < \sigma^2 < \frac{N_2 s_2^2}{\chi_1^2} \tag{8.76}$$

Choose $\alpha = 0.05$, hence, from the data in Table 8-1, there are $n_1 = N_1 - 1 = 17$ and $n_2 = N_2 - 1 = 15$ degrees of freedom, respectively. Then,

$$\chi^2_{2/d.f.=15;\ \alpha/2=0.025} = 27.488$$

and

$$\chi^2_{1/d.f.=17;\ \alpha/2=0.975} = 7.564$$

These numbers, together with those developed in Table 8-1, yield the following:

$$\frac{(18)(2.264706)}{27.488} < \sigma^2 < \frac{(16)(2.13334)}{7.564}$$

Hence,

$$1.4830 < \sigma^2 < 4.51260 \tag{8.77}$$

Thus,

$$P(1.4830 < \sigma^2 < 4.51260) = 0.95 \tag{8.78}$$

8.10.4 Confidence Limits for the Standard Deviation

For the numerical example cited above, confidence limits on the standard deviation can be found by taking the square *root* of all members of inequality (8.77). Thus,

$$1.21778 < \sigma < 2.124288 \;, \tag{8.79}$$

and likewise,

$$P(1.21778 < \sigma < 2.124288) = 0.95 \tag{8.80}$$

8.10.5 Confidence Limits for the F-Statistic

As mentioned earlier, the F-statistic, defined by equation (8.58), is used to test various hypotheses regarding *parent population* variances. The hypotheses tested include:

$$(1)\begin{cases} H_o:\ \sigma_1^2 = \sigma_2^2 \\ H_1:\ \sigma_1^2 \neq \sigma_2^2 \end{cases}$$ Test with a two-tail test.

$$(2)\begin{cases} H_o:\ \sigma_1^2 = \sigma_2^2 \\ H_o:\ \sigma_1^2 < \sigma_2^2 \end{cases}$$ Test with a left-hand tail test only.

$$(3)\begin{cases} H_o:\ \sigma_1^2 = \sigma_2^2 \\ H_1:\ \sigma_1^2 > \sigma_2^2 \end{cases}$$ Test with a right-hand tail test only.

The F-test illustrated earlier was an example of (3) above, namely testing H_o: $\sigma_1^2 = \sigma_2^2$ versus the alternative hypothesis H_1: $\sigma_1^2 > \sigma_2^2$. At the $\alpha = 0.05$ level of significance, it was found that the null hypothesis was accepted. Parenthetically, it should be noted that in this example, $A_{12} = 0$, $B_{12} = F(17, 15) = 2.372$, and $F_o = 1.0538$, so that the upper tail (right-handed tail) in the F-density function is the only critical region considered. Thus, since $0 < F_o < F_\alpha$, or numerically, $0 < 1.0538 < 2.372$, then,

$$P(0 < 1.0538 < 2.372) = 0.95 \ , \tag{8.81}$$

so that for this particular example, it can be stated with 95% confidence that the two samples shown in Table 8-1 were extracted from parent density functions having the same variance.

To generalize the above numerical results somewhat, let

$$F_o = \frac{N_1 \ s_1^2/(N_1-1)}{N_2 \ s_2^2/(N_2-1)} = \frac{N_1 \ s_1^2/n_1}{N_2 \ s_2^2/n_2} \ , \ n_1 = N_1 - 1$$

Take each case above in turn and note

$$(1) \left\{ \begin{array}{l} H_o: \ \sigma_1^2 = \sigma_2^2 \\ \\ H: \ \sigma_1^2 \neq \sigma_2^2 \\ \\ \qquad\qquad \displaystyle\int_0^{A_{12}} f(F)\ dF = \alpha/2 \\ A_{12} < F_o < B_{12} \\ \qquad\qquad \displaystyle\int_{B_{12}}^{\infty} f(F)\ dF = \alpha/2 \\ \\ P(A_{12} < F_o < B_{12}) = 1 - \alpha \end{array} \right\} \text{ a two-tail test}$$

$$
(2)\begin{cases}
H_o:\ \sigma_1^2 = \sigma_2^2 \\
H_1:\ \sigma_1^2 < \sigma_2^2 \\
A_{12} < F_o < +\infty\ ;\ \displaystyle\int_0^{A_{12}} f(F)\,dF = \alpha \\
P(A_{12} < F_o < +\infty) = 1 - \alpha
\end{cases}
$$

$$
(3)\begin{cases}
H_o:\ \sigma_1^2 = \sigma_2^2 \\
H_1:\ \sigma_1^2 > \sigma_2^2 \\
0 < F_o < B_{12}\ ;\ \displaystyle\int_{B_{12}}^{\infty} f(F)\,dF = \alpha \\
P(0 < F_o < B_{12}) = 1 - \alpha
\end{cases}
$$

For the numerical example above, data from Table 8-1 was used and case (3) above was the one under consideration. Here it should be recalled that $A_{12} = 1/B_{21}$, and $B_{12} = 1/A_{21}$.

APPENDIX B

Future Efforts Under Phases II, III and IV of
AAES Analyses

Section 9.0

PROPOSED APPLICATIONS OF PHASE I STATISTICAL ANALYSIS METHODOLOGY TO SPECIFIC AAES

APPENDIX B: FUTURE EFFORTS UNDER PHASES II, III AND IV OF AAES ANALYSES

9.0 Proposed Applications of Phase I Statistical Analysis Methodology to Specific AAES

Analyses to be applied to a specific AAES are summarized in Figures 9-1 and 9-2. The first of these proposes application of methodology developed in Phase I to a single attribute data stream. Here, an example of a single attribute data stream could be injury data versus chronological ejection history.

Use of Figure 9-1 is along the following lines: (1) Select an AAES to be studied. This selection would be made from the ESCAPAC, Martin-Baker-MK7, Rockwell International, SIIIS-3 (Stencel), or other systems under evaluation; (2) Select the single attribute data stream to be studied; (3) Re-format the data if necessary, or desirable, thereby making it more easily quantified; (4) Apply Bayesian statistics, if appropriate, to refine a priori probability estimates; (5) Perform non-parametric (distribution free) tests such as the run and trend tests. Next, subject the data to hypothesis testing: (1) Apply the binomial density function to test various dichotomous attributes the data may have; (2) test randomness of the data by hypothesizing that a random sample of size n is extracted from a normal parent density function; (3) if appropriate, test whether the data sample is extracted from a continuous parent density function which is gamma distributed; and (4) apply the t-density function to test whether the sample mean is related ("found in a given neighborhood") to a given parent density function mean. Confidence limits will be given to appropriate hypothesis tests.

In the case of dual attribute data streams, as illustrated by Figure 9-2 for a given AAES, analyses would proceed as follows: (1) Select dual attribute data streams, for example, injury versus speed, injury versus altitude, etc.; (2) if desirable, pass each attribute through the single attribute data stream analysis system described by Figure 9-1; (3) perform regression analysis on two data streams; (4) subject the dual streams to low order contingency table data analysis; (5) as appropriate, apply higher order contingency

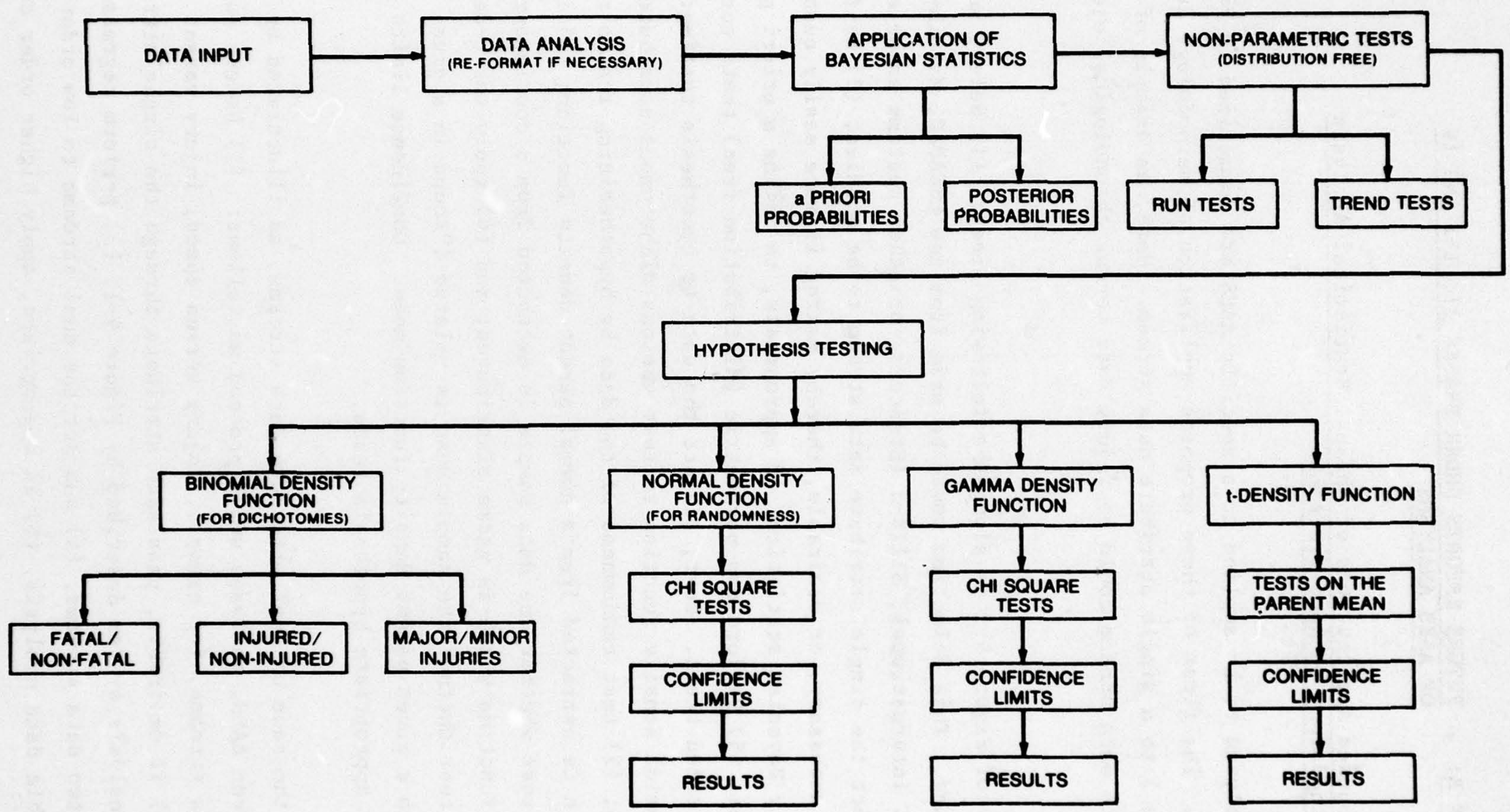

FIGURE 9-1. STATISTICAL ANALYSIS OF A SINGLE ATTRIBUTE DATA STREAM

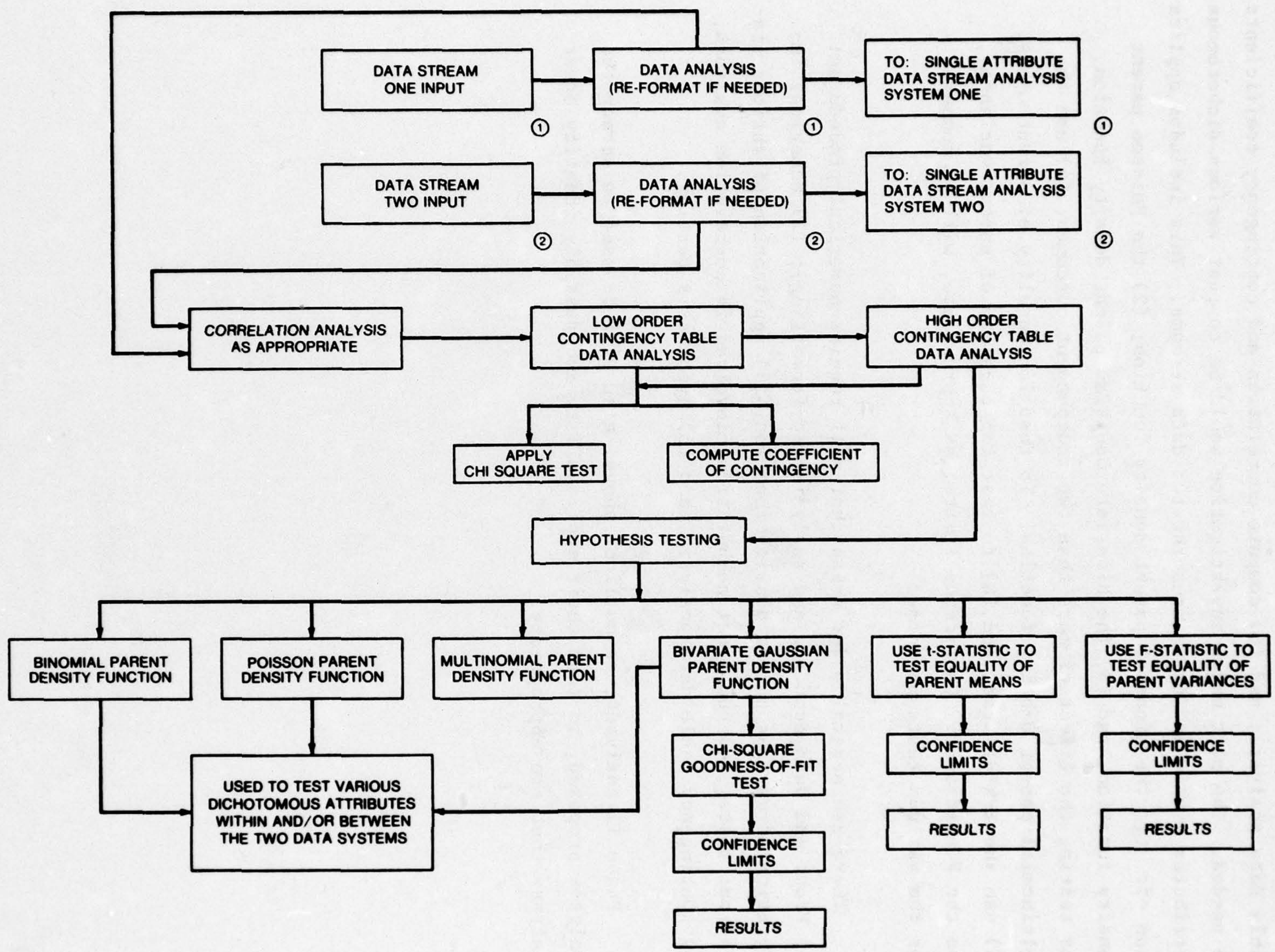

FIGURE 9-2. STATISTICAL ANALYSES OF DUAL ATTRIBUTE DATA STREAMS

table data analysis; and (6) compute correlation and contingency coefficients as needed. The next major investigation would be to test various dichotomous attributes within and/or between the two data streams. This includes application of: (1) the binomial parent density function; (2) the Poisson parent density function, and (3) the bivariate Gaussian parent density function. For testing the effect of more than two independent outcomes; (4) use the multinomial parent density function. To test for equality of parent means, (5) use the t-statistic, and (6) to test for equality of parent variances, use the F-statistic. Confidence limits, as appropriate, will be computed for the various tests performed.

There are several major areas that will require numerical procedures: (1) Front end data decoding and analysis (re-formatting); (2) numerical integration of various parent density function, (3) application of various statistical tests, (4) numerical computations involved in correlation analysis, (5) contingency table data analysis, and (6) hypothesis testing.

Phase II analyses, as outlined above, also will be used to screen the analyses proposed, to test their applicability and possibly identify other analyses that are appropriate.

Section 10.0

OUTLINE OF PHASE III ACTIVITIES

10.0 Outline of Phase III Activities

Proposed activities in Phase III, together with background information, is shown schematically in Figure 10-1.

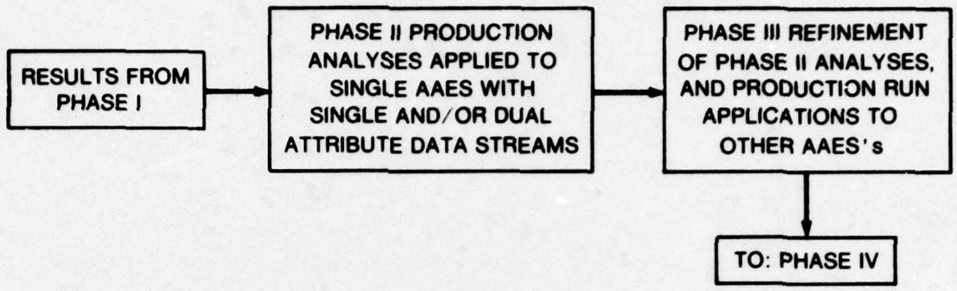

FIGURE 10-1. PROPOSED APPLICATION OF THE DERIVED STATISTICAL ANALYSIS METHODOLOGY TO MORE THAN ONE AAES.

The analyses developed and results obtained in Phase III will be used to assist designated laboratories, systems designers/manufacturers and other selected activities understand the approach, the results of the investigation, and significance of the results so obtained. Interpretations will be thoroughly presented. This is deemed appropriate in that certain corrective actions may need to be developed so that existing and/or future AAES's can be improved.

Section 11.0

SKETCH OF PHASE IV ACTIVITIES

11.0 Sketch of Phase IV Activities

Following successful completion of Phases I, II, and III, the fourth phase of this project will be structured along the following lines:

- **Program Planning and Development**

  Here, plans will be developed and executed on an annual basis. This will be a continuing review. Inputs provided for this review will include the following: (1) data for the reported year, and (2) data for the reported year and the four preceding years. Such procedures will assist a comparative evaluation of the effects of introducing AAES improvements on existing and new AAES's.

- **Awareness of Specification Modification**

  In the execution of Phases I through IV maintain cognizance of the constant need for technical specification updating. Special effort will be exerted in the area of data and test requirements. This task will be done concurrently with the specific tasks outlined above.

- **Problem Isolation**

  At all times special problems that are discovered during the accomplishment of the above tasks should be brought to the attention of NAVAIR-531.